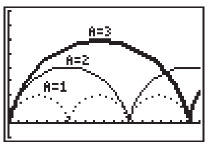
# Chapter 11

## Parametric, Vector, and Polar Functions

**Section 11.1** Parametric Functions (pp. 537–543)

**Exploration 1** Investigating Cycloids

1. 

[0, 20] by [−1, 8]

2. $x = 2na\pi$ for any integer $n$.

3. $a > 0$ and $1 - \cos t \geq 0$ so $y \geq 0$.

4. An arch is produced by one complete turn of the wheel. Thus, they are congruent.

5. The maximum value of $y$ is $2a$ and occurs when $x = (2n+1)a\pi$ for any integer $n$.

6. The function represented by the cycloid is periodic with period $2a\pi$, and each arch represents one period of the graph. In each arch, the graph is concave down, has an absolute maximum of $2a$ at the midpoint, and an absolute minimum of 0 at the two endpoints.

**Quick Review 11.1**

1. $t = x - 1$
$y = 2t + 3 = 2(x-1) + 3 = 2x + 1$

2. $t = \dfrac{x}{3}$
$y = 54t^3 - 3 = 54\left(\dfrac{x}{3}\right)^3 - 3 = 2x^3 - 3$

3. $x^2 = \sin^2 t$
$y^2 = \cos^2 t$
$\cos^2 t + \sin^2 t = 1$
$x^2 + y^2 = 1$

4. $y = \sin 2t = 2\sin t \cos t$
$y = 2x$

5. $x^2 = \tan^2 \theta$
$y^2 = \sec^2 \theta$
$\sec^2 \theta = 1 + \tan^2 x$
$y^2 = 1 + x^2$

6. $x^2 = \csc^2 \theta = 1 + \cot^2 \theta$
$y^2 = \cot^2 \theta$
$x^2 = 1 + y^2$

7. $x^2 = \cos^2 \theta$
$y = \cos 2\theta = 2\cos^2 \theta - 1$
$y = 2x^2 - 1$

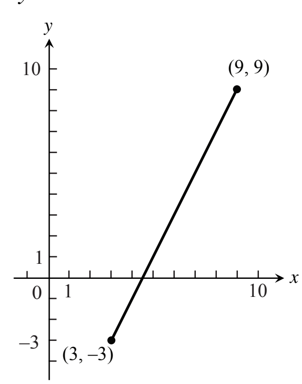
8. $x^2 = \sin^2 \theta$
$y = \cos 2\theta = 1 - 2\sin^2 \theta$
$y = 1 - 2x^2$

9. $\cos^2 \theta + \sin^2 \theta = 1$
$x^2 + y^2 = 1$
$y^2 = 1 - x^2$
$y = \sqrt{1 - x^2}$
since $y \geq 0$ for $0 \leq \theta \leq \pi$.

10. $\cos^2 \theta + \sin^2 \theta = 1$
$x^2 + y^2 = 1$
$y^2 = 1 - x^2$
$y = -\sqrt{1 - x^2}$
since $y \leq 0$ for $\pi \leq \theta \leq 2\pi$.

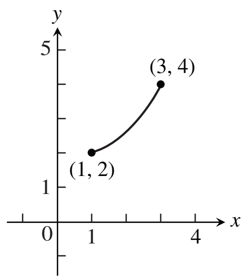
**Section 11.1** Exercises

1. Yes, $y$ is a function of $x$.
$y = 2x - 9$



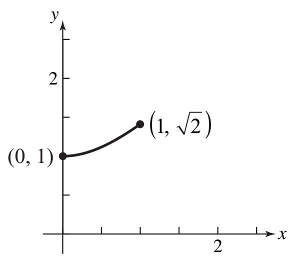
**2.** Yes, $y$ is a function of $x$.

$$y = \frac{x^2+7}{4}$$



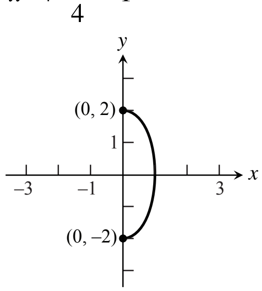

**3.** Yes, $y$ is a function of $x$.

$$y = \sqrt{x^2+1}$$



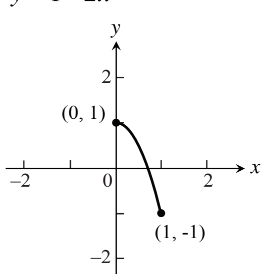

**4.** No, $y$ is not a function of $x$.

$$x^2 + \frac{y^2}{4} = 1$$



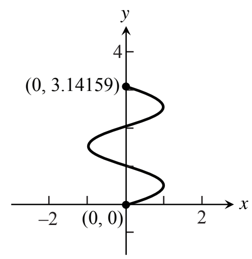

**5.** Yes, $y$ is a function of $x$.

$$y = 1 - 2x^2$$




**6.** No, $y$ is not a function of $x$.

$$x = \sin(3y)$$




**7.** **(a)** $$\frac{dy}{dx} = \frac{-2\sin t}{4\cos t} = -\frac{1}{2}\tan t$$

**(b)** $$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(-\frac{1}{2}\tan t\right)}{4\cos t} = \frac{-\frac{1}{2}\sec^2 t}{4\cos t} = -\frac{1}{8}\sec^3 t$$

**8.** **(a)** $$\frac{dy}{dx} = \frac{-\sqrt{3}\sin t}{-\sin t} = \sqrt{3}$$

**(b)** $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\sqrt{3}\right)}{-\sin t} = 0$$

**9.** **(a)** $$\frac{dy}{dx} = \frac{\frac{3}{2}(3t)^{-1/2}}{-\left(\frac{1}{2}\right)(t+1)^{-1/2}} = -\sqrt{\frac{9t+9}{3t}} = -\sqrt{3+\frac{3}{t}}$$

**(b)** $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(-\sqrt{3+\frac{3}{t}}\right)}{-\left(\frac{1}{2}\right)(t+1)^{-1/2}} = -\frac{\sqrt{3}}{t^{3/2}}$$

**10.** **(a)** $$\frac{dy}{dx} = \frac{\frac{1}{t}}{-\frac{1}{t^2}} = -t$$

**(b)** $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(-t)}{-\frac{1}{t^2}} = t^2$$

**11. (a)** $\frac{dy}{dx} = \frac{3t^2}{2t-3}$

**(b)** $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{3t^2}{2t-3}\right)}{2t-3} = \frac{6t^2-18t}{(2t-3)^3}$

**12. (a)** $\frac{dy}{dx} = \frac{2t-1}{2t+1}$

**(b)** $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{2t-1}{2t+1}\right)}{2t+1} = \frac{4}{(2t+1)^3}$

**13. (a)** $\frac{dy}{dx} = \frac{\sec t \tan t}{\sec^2 t} = \sin t$

**(b)** $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\sin t}{\sec^2 t} = \cos^3 t$

**14. (a)** $\frac{dy}{dx} = \frac{-2\sin(2t)}{-2\sin t} = 2\cos t$

**(b)** $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(2\cos t)}{-2\sin t} = 1$

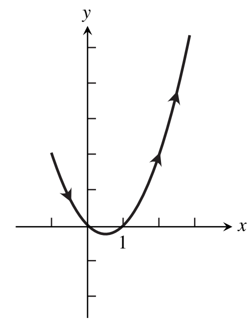
**15. (a)** $\frac{dy}{dx} = \frac{4\left(\frac{1}{t}\right)}{\frac{1}{t}} = 4$

**(b)** $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(4)}{\frac{1}{t}} = 0$

**16. (a)** $\frac{dy}{dx} = \frac{5e^{5t}}{\frac{1}{t}} = 5te^{5t}$

**(b)** $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(5te^{5t})}{\frac{1}{t}} = 25t^2e^{5t} + 5te^{5t}$

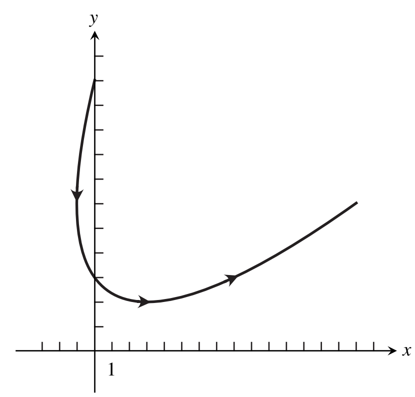
**17. (a)**




**(b)** (0.5, –0.25)

**(c)** We seek to minimize $y$ as a function of $t$, so we compute $\frac{dy}{dt} = 2t+1$, which is negative for $-2 \le t < -0.5$ and positive for $-0.5 < t \le 2$. There is a relative minimum at $t = -0.5$, where $(x, y) = (0.5, -0.25)$.

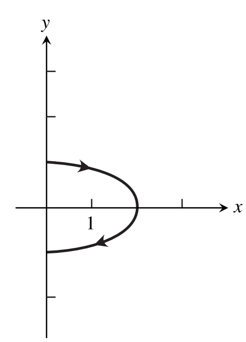
**18. (a)**




**(b)** (–1, 6)

**(c)** We seek to minimize $x$ as a function of $t$, so we compute $\frac{dx}{dt} = 2t+2$, which is negative for $-2 \le t < -1$ and positive for $-1 < t \le 3$. There is a relative minimum at $t = -1$, where $(x, y) = (-1, 6)$.

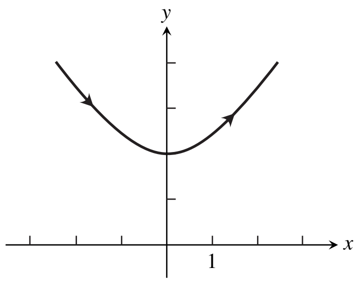
**19. (a)**




**(b)** (2, 0)

**(c)** We seek to maximize $x$ as a function of $t$, so we compute $\frac{dx}{dt} = 2\cos t$, which is positive for $0 \le t < \frac{\pi}{2}$ and negative for $\frac{\pi}{2} < t \le \pi$. There is a relative maximum at $t = \frac{\pi}{2}$, where $(x, y) = (2, 0)$.

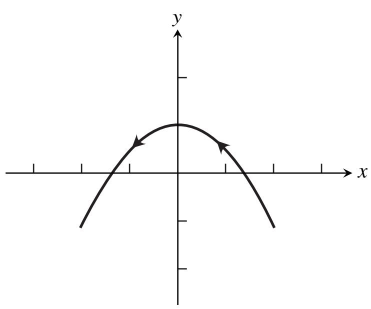
**20.** **(a)**




**(b)** (0, 2)

**(c)** We seek to minimize $y$ as a function of $t$, so we compute $\frac{dy}{dt} = 2\sec t \tan t$, which is negative for $-1 \le t < 0$ and positive for $0 < t \le 1$. There is a relative minimum at $t = 0$, where $(x, y) = (0, 2)$.

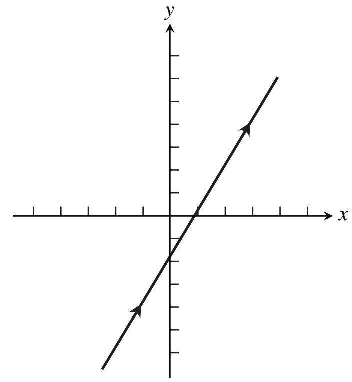
**21.** **(a)**




**(b)** (0, 1)

**(c)** We seek to maximize $y$ as a function of $t$, so we compute $\frac{dy}{dt} = -2\sin(2t)$, which is positive for $1.5 \le t < \pi$ and negative for $\pi < t \le 4.5$. There is a relative maximum at $t = \pi$, where $(x, y) = (0, 1)$.

**22.** **(a)**




**(b)** $(\ln(50), \ln(400)) \approx (3.912, 5.991)$

**(c)** We seek to maximize $x$ as a function of $t$, so we compute $\frac{dx}{dt} = \frac{1}{t}$, which is positive for all $t > 0$. There is an endpoint maximum at $t = 10$, where $(x, y) = (\ln(50), \ln(400))$.

**23.** $\frac{dy}{dt} = \frac{d}{dt}(-1 + \sin t) = \cos t$ and
$\frac{dx}{dt} = \frac{d}{dt}(2 + \cos t) = -\sin t$.

**(a)** Tangent is horizontal when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \ne 0$.
If $\frac{dy}{dt} = 0$, then $\cos t = 0$, and
$x = 2 + \cos t = 2$.
If $\cos t = 0$, then $\sin t = \pm 1$, and
$y = -1 + \sin t = 0$ or $-2$.
The points are $(2, 0)$ and $(2, -2)$.

**(b)** Tangent is vertical when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \ne 0$. If $\frac{dx}{dt} = 0$, then $\sin t = 0$, and
$y = -1 + \sin t = -1$.
If $\sin t = 0$, then $\cos t = \pm 1$, and
$x = 2 + \cos t = 1$ or $3$.
The points are $(1, -1)$ and $(3, -1)$.

**24.** $\frac{dy}{dt} = \frac{d}{dt}(\tan t) = \sec^2 t$ and
$\frac{dx}{dt} = \frac{d}{dt}(\sec t) = \sec t \tan t.$

**(a)** Tangent is horizontal when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0.$ Since $\frac{dy}{dt} = \sec^2 t > 0$ for all $t$, there are no points where the tangent line is horizontal.

**(b)** Tangent is vertical when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0.$ If $\frac{dx}{dt} = 0,$ then $\sec t \tan t = 0,$ so $y = \tan t = 0.$
If $\tan t = 0,$ then $x = \sec t = \pm 1.$
The points are (1, 0) and (–1, 0).

**25.** $\frac{dy}{dt} = \frac{d}{dt}(t^3 - 4t) = 3t^2 - 4$ and
$\frac{dx}{dt} = \frac{d}{dt}(2 - t) = -1.$

**(a)** If $\frac{dy}{dt} = 0,$ then $3t^2 - 4 = 0$ and $t = \pm\frac{2}{\sqrt{3}}.$ Using the parametric formulas for $x$ and $y$, the points are
$\left(2 - \frac{2}{\sqrt{3}}, -\frac{16\sqrt{3}}{9}\right)$ and $\left(2 + \frac{2}{\sqrt{3}}, \frac{16\sqrt{3}}{9}\right).$
(In decimal form, they are (0.845, –3.079) and (3.155, 3.079).)

**(b)** Tangent is vertical when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0.$ Since $\frac{dx}{dt} = -1 \neq 0$ for all $x$, there are no points where the tangent line is vertical.

**26.** $\frac{dy}{dt} = \frac{d}{dt}(1 + 3\sin t) = 3\cos t$ and
$\frac{dx}{dt} = \frac{d}{dt}(-2 + 3\cos t) = -3\sin t.$

**(a)** If $\frac{dy}{dt} = 0$, then $\cos t = 0$ and
$x = -2 + 3 \cos t = -2.$
If $\cos t = 0$, then $\sin t = \pm 1$, and
$y = 1 + 3 \sin t = -2$ or 4.
The points are (–2, –2) and (–2, 4).

**(b)** Tangent is vertical when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0.$ If $\frac{dx}{dt} = 0$, then $\sin t = 0$, and
$y = -2 + 3 \cos t = -2.$
If $\sin t = 0$, then $\cos t = \pm 1$, and
$x = -2 + 3 \cos t = -5$ or 1.
The points are (–5, 1) and (1, 1).

**27.** $S = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt$

$S = \int_0^{2\pi} \sqrt{1}\, dt$

$S = t\Big|_0^{2\pi} = 2\pi - 0$

$S = 2\pi$

**28.** $S = \int_0^{\pi} \sqrt{(3\cos t)^2 + (-3\sin t)^2}\, dt$

$S = \int_0^{\pi} \sqrt{9}\, dt$

$S = 3t\Big|_0^{\pi} = 3\pi - 0$

$S = 3\pi$

**29.** $S=\int_0^{\pi/2}((-8\sin t+8\sin t+8t\cos t)^2+(8\cos t-8\cos t+8t\sin t)^2)^{1/2}\,dt$

$S=\int_0^{\pi/2}((8t\cos t)^2+(8t\sin t)^2)^{1/2}dt$

$S=\int_0^{\pi/2}8t\,dt$

$S=4t^2\Big|_0^{\pi/2}=\pi^2-0$

$S=\pi^2$

**30.** $\int_0^{2\pi}([-6\cos^2(t)\sin(t)]^2+[6\sin^2(t)\cos(t)]^2)^{1/2}\,dt$

$=\int_0^{2\pi}6\sqrt{\cos^4(t)\sin^2(t)+\sin^4(t)\cos^2(t)}\;dt$

$=\int_0^{2\pi}6\sqrt{\{\cos^2(t)+\sin^2(t)\}[\cos^2(t)\sin^2(t)\}}\;dt$

$=\int_0^{2\pi}6\sqrt{\cos^2(t)\sin^2(t)}\;dt$

$=4\cdot\int_0^{\pi/2}6\cos t\,\sin t\;dt$

$=4\cdot\int_0^{\pi/2}3\cdot\sin 2t\;dt$

$=12\left\{\frac{-\cos 2t}{2}\right\}\Bigg|_0^{\pi/2}$

$=6\cdot\;[1-(-1)]$

$=12$

**31.** $S=\int_0^3\left(\left(\sqrt{2t+3}\right)^2+(1+t)^2\right)^{1/2}dt$

$S=\int_0^3(t^2+4t+4)^{1/2}dt$

$S=\frac{21}{2}$

**32.** $S=\int_0^2\left(\left(\sqrt{8t+8}\right)^2+(2t+1)^2\right)^{1/2}dt$

$S=\int_0^2(4t^2+12t+9)^{1/2}\,dt$

$S=10$

**33.** $S=\int_0^1((t^2)^2+(t)^2)^{1/2}dt$

$S=\int_0^1(t^4+t^2)^{1/2}\,dt$

$S=\frac{2\sqrt{2}-1}{3}$

**34.** $S = \int_0^{\pi/3} \sqrt{(\sec t - \cos t)^2 + (-\sin t)^2}\ dt$

$= \int_0^{\pi/3} \sqrt{\tan^2 t}\ dt$

$= \int_0^{\pi/3} \tan\ x\ dx$

$= \left[-\ln\left|\cos x\right|\right]_0^{\pi/3}$

$= \ln 2$

**35.** **(a)** $x' = -2\sin 2t,\ y' = 2\cos 2t,$ so

Length $= \int_0^{\pi/2} \sqrt{(-2\sin 2t)^2 + (2\cos 2t)^2}$

$= \int_0^{\pi/2} 2\ dt$

$= \pi.$

**(b)** $x' = \pi\cos \pi t,\ y' = -\pi \sin \pi t,$ so

Length $= \int_{-1/2}^{1/2} \sqrt{(\pi\cos \pi t)^2 + (-\pi \sin \pi t)^2}\, dt$

$= \int_{-1/2}^{1/2} \pi\ dt$

$= \pi.$

**36.** $x' = -3\sin t,\ y' = 4\cos t,$ so Length $= \int_0^{2\pi} \sqrt{(-3\ \sin t)^2 + (4\ \cos t)^2}\, dt$ which using NINT evaluates to $\approx$ 22.103.

**37.** In the first integral, replace $t$ with $x$. Then $\frac{dx}{dt}$ becomes $\frac{dx}{dx} = 1.$

**38.** Parameterize the curve as $x = g(y)$, $y = y$, $c \le y \le d$. The parameter is $y$ itself, so replace $t$ with $y$ in the general formula. Then $\frac{dy}{dt}$ becomes $\frac{dy}{dy} = 1.$

**39.** $\frac{dx}{dt} = a(1 - \cos t)$

(Note: integrate with respect to $x$ from 0 to $2a\pi$; integrate with respect to $t$ from 0 to $2\pi$.)

Area $= \int_0^{2a\pi} y\ dx$

$= \int_0^{2\pi} a(1 - \cos t)a(1 - \cos t)\ dt$

$= a^2 \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\ dt$

$= a^2 \left[t - 2\sin t + \frac{t}{2} + \frac{1}{4}\sin 2t\right]_0^{2\pi}$

$= 3\pi a^2$

**40.** $\frac{dx}{dt} = a(1-\cos t)$, so

$$\text{Volume } = \int_0^{2\pi} \pi[a(1-\cos t)]^2 a(1-\cos t)dt$$
$$= \pi a^3 \int_0^{2\pi} (1-3\ \cos t + 3\ \cos^2 t - \cos^3 t)dt$$
$$= \pi a^3 \left[ t - 3\ \sin t + \frac{3}{2}t + \frac{3}{4}\sin\ 2t - \left( \sin t - \frac{1}{3}\sin^3 t \right) \right]_0^{2\pi}$$
$$= 5\pi^2 a^3$$

**41.** $x = at - b\sin t$ and
$y = a - b\ \cos t\ (0 < b < a)$

**42.** $x = at - b\sin t$ and
$y = a - b\ \cos t\ (a < b < 2a)$

**43.** $S = \int_0^{2\pi} ((3-2\cos t)^2 + (2\sin t)^2)^{1/2}\,dt$
$S = \int_0^{2\pi} (9 - 12\cos t + 4\cos^2 t + 4\sin^2 t)^{1/2}\,dt$
$S = \int_0^{2\pi} (13 - 12\cos t)^{1/2}\,dt$
$S \approx 21.010$

**44.** $S = \int_0^{2\pi} ((2-3\cos t)^2 + (3\sin t))^{1/2}\,dt$
$S = \int_0^{2\pi} (4 - 12\cos t + 9\cos^2 t + 9\sin^2 t)^{1/2}\,dt$
$S = \int_0^{2\pi} (13 - 12\ \cos t)^{1/2}\,dt$
$S \approx 21.010$

**45.** False. Indeed, $y$ may not even be a function of $x$. (See Example 1.)

**46.** True. The ordered pairs $(x,\ f(x))$ and $(t,\ f(t))$ are exactly the same.

**47.** B

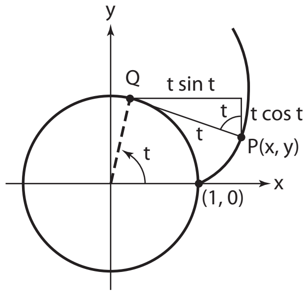
**48.** C; if $x = \sin t$ and $y = \csc t$, then we can eliminate the parameter to get $y = \frac{1}{x}$. Since sin $t$ and csc $t$ are both positive for $0 < t < \frac{\pi}{2}$, the path follows a portion of the curve in the first quadrant, where $y = \frac{1}{x}$ is decreasing and concave up.

**49.** C; $x = \ln(t) = \ln(y)$
$y = e^x$

**50.** D

**51.** **(a)** $\overline{QP}$ has length $t$, so $P$ can be obtained by starting at $Q$ and moving $t\sin t$ units right and $t\cos t$ units downward.
(If either quantity is negative, the corresponding direction is reversed.) Since $Q = (\cos t, \sin t)$, the

coordinates of $P$ are $x = \cos t + t \sin t$ and $y = \sin t - t \cos t$.




**(b)** $x' = t\cos t,\ y' = t\sin t,$ so

$$\begin{aligned}\text{Length} &= \int_0^{2\pi} \sqrt{(t\cos t)^2 + (t\sin t)^2}\,dt \\ &= \int_0^{2\pi} t\,dt \\ &= 2\pi^2\end{aligned}$$

**52.** All distances are $a$ times as big as before.

**(a)** $x = a(\cot t + t\sin t),\ y = a(\sin t - t\cos t)$

**(b)** $\text{Length} = 2a\pi^2$

For exercises 53–56, $x' = v_0 \cos\theta$ and $y' = v_0 \sin\theta - 32t,$ and $y = 0$ for $t = 0$ or $t = \dfrac{v_0 \sin\theta}{16}$. The maximum height is attained in mid-flight at $t = \dfrac{v_0 \sin\theta}{32}$. To find the path length, evaluate $\int_0^{v_0 \sin\theta/16} \sqrt{(v_0\cos\theta)^2 + (v_0 \sin\theta - 32t)^2}\,dt$ using NINT. To find the maximum height, calculate $y_{\max} = (v_0 \sin\theta)\left(\dfrac{v_0 \sin\theta}{32}\right) - 16\left(\dfrac{v_0\sin\theta}{32}\right)^2$.

**53. (a)** The projectile hits the ground when $y = 0$.

$y = t(150\ \sin 20° - 16t) = 0$

$t = 0$ or $t = \dfrac{75}{8}\ \sin 20° \approx 3.206$

$x' = 150\ \cos 20°,\ y' = 150\ \sin 20° - 32t$

$\text{Length} = \int_0^{(75\sin 20°)/8} \sqrt{(150\ \cos 20°)^2 + (150\ \sin 20° - 32t)^2}\ dt$ which, using NINT, evaluates to $\approx 461.749$ ft

**(b)** The maximum height of the projectile occurs when $y' = 0,$ so $t = \dfrac{75}{16}\sin 20°,\ \ y\left(\dfrac{75}{16}\sin 20°\right) \approx 41.125$ ft

**54. (a)** $\approx 641.236$ ft

**(b)** $\dfrac{5625}{64} \approx 87.891$ ft

**55. (a)** $\approx 840.421$ ft

**(b)** $\frac{16,875}{64} \approx 263.672$ ft

**56. (a)** It is not necessary to use NINT.

$$\text{Length} = \int_0^{75/8} (150 - 32t)\, dt = [150t - 16t^2]_0^{75/8} = \frac{5625}{8} = 703.125 \text{ ft}$$

**(b)** $\frac{5625}{16} = 351.5625$ ft

**57.** $S = \int_0^{2\pi} 2\pi y\, ((-\sin t)^2 + \cos t^2)^{1/2}\, dt$

$$S = \int_0^{2\pi} 2\pi y\,(1)\, dt = \int_0^{2\pi} 2\pi(2 + \sin t)\, dt = 2\pi(2t - \cos t)\Big|_0^{2\pi} = 8\pi^2$$

**58.** $S = \int_0^2 2\pi y\, ((t^{-1/2})^2 + (t^{1/2})^2)^{1/2}\, dt$

$$S = \int_0^2 2\pi y \left(\frac{1}{t} + t\right)^{1/2} dt = \int_0^2 2\pi \cdot \frac{2}{3} t^{3/2} \left(\frac{1}{t} + t\right)^{1/2} dt = \int_0^2 \frac{4\pi}{3} \left(t^3 \left[\frac{1}{t} + t\right]\right)^{1/2} dt = \int_0^2 \frac{4\pi}{3} t(1 + t^2)^{1/2}\, dt \approx 14.214$$

**59.** $S = \int_0^3 2\pi y\, ((2t)^2 + (1)^2)^{1/2}\, dt$

$$S = \int_0^3 2\pi y\, (4t^2 + 1)^{1/2}\, dt = \int_0^3 2\pi(t + 1)(4t^2 + 1)^{1/2}\, dt = 178.561$$

**60.** $S = \int_0^{\pi/3} 2\pi y((\sec t - \cos t)^2 + (-\sin t)^2)^{1/2}\, dt$

$$S = \int_0^{\pi/3} 2\pi \cos t(\sec^2 t - 1)^{1/2}\, dt = \int_0^{\pi/3} 2\pi \cos t \tan t\ dt = 2\pi \int_0^{\pi/3} \sin t\ dt = 2\pi(-\cos t)\Big|_0^{\pi/3} = 2\pi\left(-\frac{1}{2} + 1\right) = \pi$$

## Section 11.2 Vectors in the Plane (pp. 544–554)

### Quick Review

**1.** $\sqrt{(5-1)^2 + (3-2)^2} = \sqrt{17}$

**2.** $\frac{3-2}{5-1} = \frac{1}{4}$

**3.** Solve $\frac{3-b}{5-3} = \frac{1}{4}$; $b = \frac{5}{2}$.

**4.** Slope of $\overline{PQ} = -\frac{1}{\overline{RQ}}$, so

$\frac{3-2}{5-1} = -\frac{5-3}{3-b}$ and $b = 11$.

**5.** Slope of $\overline{AB}$ = Slope of $\overline{CD}$, so $\frac{3-0}{1-0} = \frac{3-0}{5-a}$ and $a = 4$.

**6.** Slope of $\overline{AB}$ = Slope of $\overline{CD}$, so $\frac{5-1}{3-1} = \frac{2-b}{6-8}$ and $b = 6$.

**7.** $v(t) = \frac{d}{dt}(t \sin t)$

$$v(t) = \sin t + t \cos t$$
$$a(t) = \frac{d}{dt}(\sin t + t \cos t)$$
$$a(t) = 2\cos t - t \sin t$$

**8.** $x(t) = \int v(t)dt = \int (3t^2 - 12t)dt$

$$x(t) = t^3 - 6t^2 + C$$
$$x(0) = 0^3 - 6(0)^2 + C = 40$$
$$C = 40$$
$$x(4) = 4^3 - 6(4)^2 + 40 = 8$$

**9.** $|x(t)| = \left|\int_0^4 v(t)dt\right| = \left|\int_0^4 (3t^2 - 12t)dt\right|$

$|x(t)| = \left|\left[t^3 - 6t^2\right]_0^4\right| = 32$

**10.** $\int_0^{2\pi} \sqrt{(2\cos 2t)^2 + (-3\sin 3t)^2}\, dt$

$= \int_0^{2\pi} \sqrt{4\cos^2 2t + 9\sin^2 3t}\ dt$

$= 15.289$

**Section 11.2** Exercises

**1.** $(2, 3) - (0, 0) = \langle 2, 3\rangle$

**2.** $(0, 0) - (2, 3) = \langle -2, -3\rangle$

**3.** $(2, -1) - (1, 3) = \langle 1, -4\rangle$

**4.** $P = \dfrac{-4+2}{2}, \dfrac{3-1}{2} = (-1, 1)$

$(-1, 1) - (0, 0) = \langle -1, 1\rangle$

**5.** $|v| = \sqrt{2^2 + 2^2} = \sqrt{8}$

$\tan\theta = \dfrac{\sqrt{2}}{2}, \theta = 45°$

**6.** $|v| = \sqrt{\left(-\sqrt{2}\right)^2 + \sqrt{2}^2} = 2$

$\tan\theta = -\dfrac{\sqrt{2}}{\sqrt{2}}, \ \theta = 135°$

**7.** $|v| = \sqrt{\sqrt{3}^2 + 1^2} = 2$

$\tan\theta = \dfrac{1}{\sqrt{3}}, \theta = 30°$

**8.** $|v| = \sqrt{-2^2 + \left(-2\sqrt{3}\right)^2} = 4$

$\tan\theta = \dfrac{-2\sqrt{3}}{2}, \theta = 240°$

**9.** $|v| = \sqrt{(-5)^2 + 0^2} = 5$

$\tan\theta = \dfrac{0}{-5}, \ \theta = 180°$

**10.** $|v| = \sqrt{0^2 + 4^2} = 4$

$\cos\theta = \dfrac{0}{4}, \ \theta = 90°$

**11.** $x = 4\cos(180°) = -4$

$y = 4\sin(180°) = 0$

$\langle -4, 0\rangle$

**12.** $x = 6\cos(270°) = 0$

$y = 6\sin(270°) = -6$

$\langle 0, -6\rangle$

**13.** $x = 5\cos(100°) \approx -0.868$

$y = 5\sin(100°) \approx 4.924$

$\langle -0.868, 4.924\rangle$

**14.** $x = 13\cos(200°) \approx -12.216$

$y = 13\sin(200°) \approx -4.446$

$\langle -12.216, -4.446\rangle$

**15.** $x = 3\sqrt{2}\cos\left(\dfrac{180}{\pi}\dfrac{\pi}{4}\right) = 3$

$y = 3\sqrt{2}\sin\left(\dfrac{180}{\pi}\dfrac{\pi}{4}\right) = 3$

$\langle 3, 3\rangle$

**16.** $x = 2\sqrt{3}\cos\left(\dfrac{180}{\pi}\dfrac{\pi}{6}\right) = 3$

$y = 2\sqrt{3}\sin\left(\dfrac{180}{\pi}\dfrac{\pi}{6}\right) = \sqrt{3}$

$\langle 3, \sqrt{3}\rangle$

**17.** **(a)** $\langle 3(3), 3(-2)\rangle = \langle 9, -6\rangle$

**(b)** $\sqrt{9^2 + (-6)^2} = \sqrt{117} = 3\sqrt{13}$

**18.** **(a)** $\langle -2(-2), -2(5)\rangle = \langle 4, -10\rangle$

**(b)** $\sqrt{4^2 + (-10)^2} = \sqrt{116} = 2\sqrt{29}$

**19.** **(a)** $\langle 3 + (-2), -2 + 5\rangle = \langle 1, 3\rangle$

**(b)** $\sqrt{1^2 + 3^2} = \sqrt{10}$

**20.** **(a)** $\langle 3 - (-2), -2 - 5\rangle = \langle 5, -7\rangle$

**(b)** $\sqrt{5^2+(-7)^2}=\sqrt{74}$

**21.** **(a)** $2\mathbf{u}=\langle 2(3), 2(-2)\rangle=\langle 6,-4\rangle$
$3\mathbf{v}=\langle 3\,(-2),\ 3(5)\rangle=\langle -6, 15\rangle$
$2\mathbf{u}-3\mathbf{v}=\langle 6-(-6),\ -4-15\rangle=\langle 12,\ -19\rangle$

**(b)** $\sqrt{12^2+(-19)^2}=\sqrt{505}$

**22.** **(a)** $-2\mathbf{u}=\left\langle -2\,(3),\ -2\,(-2)\right\rangle=\left\langle -6,\ 4\right\rangle$
$5\mathbf{v}=\left\langle 5\ (-2),\ 5\ (5)\right\rangle=\left\langle -10,\ 25\right\rangle$
$-2\mathbf{u}+5\mathbf{v}=\left\langle -6+(-10),\ 4+25\right\rangle$
$=\left\langle -16,\ 29\right\rangle$

**(b)** $\sqrt{(-16)^2+29^2}=\sqrt{1097}$

**23.** **(a)** $\frac{3}{5}\mathbf{u}=\left\langle \frac{3}{5}(3),\ \frac{3}{5}(-2)\right\rangle=\left\langle \frac{9}{5},\ -\frac{6}{5}\right\rangle$
$\frac{4}{5}\mathbf{v}=\left\langle \frac{4}{5}(-2),\ \frac{4}{5}(5)\right\rangle=\left\langle -\frac{8}{5},\ 4\right\rangle$
$\frac{3}{5}\mathbf{u}+\frac{4}{5}\mathbf{v}=\left\langle \frac{9}{5}+\left(-\frac{8}{5}\right),\ -\frac{6}{5}+4\right\rangle$
$=\left\langle \frac{1}{5},\ \frac{14}{5}\right\rangle$

**(b)** $\sqrt{\left(\frac{1}{5}\right)^2+\left(\frac{14}{5}\right)^2}=\frac{\sqrt{197}}{5}$

**24.** **(a)** $-\frac{5}{13}\mathbf{u}=\left\langle -\frac{5}{13}(3),\ -\frac{5}{13}(-2)\right\rangle$
$=\left\langle -\frac{15}{13},\ \frac{10}{13}\right\rangle$
$\frac{12}{13}\mathbf{v}=\left\langle \frac{12}{13}(-2),\ \frac{12}{13}(5)\right\rangle=\left\langle -\frac{24}{13},\ \frac{60}{13}\right\rangle$
$-\frac{5}{13}\mathbf{u}+\frac{12}{13}\mathrm{v}=\left\langle -\frac{15}{13}+\left(-\frac{24}{13}\right), \frac{10}{13}+\frac{60}{13}\right\rangle$
$=\left\langle -3,\ \frac{70}{13}\right\rangle$

**(b)** $\sqrt{(-3)^2+\left(\frac{70}{13}\right)^2}=\frac{\sqrt{6421}}{13}$

**25.** Initial velocity is 70° north of east:
$325\left\langle \cos 70°,\ \sin 70°\right\rangle\approx\left\langle 111.157,\ 305.400\right\rangle$.
Wind velocity is 130° north of east:
$40\left\langle \cos 130°,\ \sin 130°\right\rangle\approx\left\langle -25.712,\ 30.642\right\rangle$.
Add the two vectors to get
$\approx\left\langle 85.445,\ 336.042\right\rangle$.
The speed is the magnitude, $\approx 346.735$ mph.
The direction is $\tan^{-1}\left(\frac{336.042}{85.445}\right)\approx 75.734°$
north of east, or $\approx 14.266°$ east of north.

**26.** $x=4\cos 135=-2\sqrt{2}$
$y=4\sin 135=2\sqrt{2}$
The true velocity is $\left\langle 2-2\sqrt{2},\ 2\sqrt{2}\right\rangle$, so the
true angle is $\theta=\cos^{-1}\left(\frac{2-2\sqrt{2}}{4}\right)\approx 106.3°$
and the true speed is
$\sqrt{(2-2\sqrt{2})^2+(2\sqrt{2})^2}\approx 2.95$ mph

**27.** $\mathbf{v}(t)=\frac{d\mathbf{r}(t)}{dt}=\frac{d}{dt}\left\langle 3t^2,\ 2t^3\right\rangle=\left\langle 6t,\ 6t^2\right\rangle$
$\mathbf{a}(t)=\frac{d\mathbf{v}(t)}{dt}=\frac{d}{dt}\left\langle 6t, 6t^2\right\rangle=\left\langle 6, 12t\right\rangle$

**28.** $\mathbf{v}(t)=\frac{d\mathbf{r}(t)}{dt}$
$=\frac{d}{dt}\langle \sin 2t, 2\cos t\rangle$
$=\langle 2\cos 2t, -2\sin t\rangle$
$\mathbf{a}(t)=\frac{d\mathbf{v}(t)}{dt}$
$=\frac{d}{dt}\langle 2\cos 2t, -2\sin t\rangle$
$=\langle -4\sin 2t, -2\cos t\rangle$

**29.** $\mathbf{v}(t)=\frac{d\mathbf{r}(t)}{dt}$
$=\frac{d}{dt}\left\langle te^{-t},\ e^{-t}\right\rangle$
$=\left\langle e^{-t}-te^{-t}, -e^{-t}\right\rangle$
$\mathbf{a}(t)=\frac{d\mathbf{v}(t)}{dt}$
$=\frac{d}{dt}\left\langle e^{-t}-te^{-t},\ -e^{-t}\right\rangle$
$=\left\langle -2e^{-t}+te^{-t},\ e^{-t}\right\rangle$

**30.** $\mathbf{v}(t) = \dfrac{d\mathbf{r}(t)}{dt}$

$= \dfrac{d}{dt}\langle 2\cos 3t,\ 2\sin 4t\rangle$

$= \langle -6\sin 3t,\ 8\cos 4t\rangle$

$\mathbf{a}(t) = \dfrac{d\mathbf{v}(t)}{dt}$

$= \dfrac{d}{dt}\langle -6\sin 3t,\ 8\cos 4t\rangle$

$= \langle -18\cos 3t,\ -32\sin 4t\rangle$

**31.** $\mathbf{v}(t) = \dfrac{d\mathbf{r}(t)}{dt}$

$= \dfrac{d}{dt}\langle t^2 + \sin 2t,\ t^2 - \cos 2t\rangle$

$= \langle 2t + 2\cos 2t,\ 2t + 2\sin 2t\rangle$

$\mathbf{a}(t) = \dfrac{d\mathbf{v}(t)}{dt}$

$= \dfrac{d}{dt}\langle 2t + 2\cos 2t,\ 2t + 2\sin 2t\rangle$

$= \langle 2 - 4\sin 2t,\ 2 + 4\cos 2t\rangle$

**32.** $\mathbf{v}(t) = \dfrac{d\mathbf{r}(t)}{dt}$

$= \dfrac{d}{dt}\langle t\sin t,\ t\cos t\rangle$

$= \langle \sin t + t\cos t,\ \cos t - t\sin t\rangle$

$\mathbf{a}(t) = \dfrac{d\mathbf{v}(t)}{dt}$

$= \dfrac{d}{dt}\langle \sin t + t\cos t,\ \cos t - t\sin t\rangle$

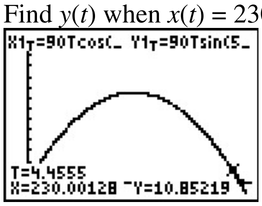
$= \langle 2\cos t - t\sin t,\ -2\sin t - t\cos t\rangle$

**33.** **(a)** The position vector of the ball at any time $t$ is $\mathbf{r}(t) = \langle x(t),\ y(t)\rangle$, where $x(t) = 90t\cos 55°$ and $y(t) = 90t\sin 55° - 16t^2$.

**(b)** $\mathbf{v}(t) = \left\langle \dfrac{dx}{dt},\ \dfrac{dy}{dt}\right\rangle$

$= \langle 90\cos 55°,\ 90\sin 55° - 32t\rangle$

**(c)** Find $y(t)$ when $x(t) = 230$.




[–20, 250] by [–10, 150]
$0 \le t \le 5$

No, the hit does not clear the 20-foot fence.

**(d)** See the graph in part (c). The ball hits the fence after about 4.4555 seconds.

**(e)** Evaluate $|\mathbf{v}(t)|$ at $t = 4.4555$.

$$|\mathbf{v}(t)| = \sqrt{(90\cos 55°)^2 + (90\sin 55° - 32t)^2}$$
$$|\mathbf{v}(4.4555)| = \sqrt{(90\cos 55°)^2 + (90\sin 55° - 32\cdot 4.4555)^2}$$
$$\approx 86.055 \text{ ft/sec}$$

**34. (a)** The position vector of the ball at any time $t$ is $\mathbf{r}(t) = \langle x(t),\ y(t)\rangle$, where $x(t) = 90 + 81t\cos 57°$ and $y(t) = 81t\sin 57° - 16t^2$.

Note that $\langle 0,\ 0\rangle$ is on the ground at the punters goal line, 30 yards = 90 feet from the kicking location.

**(b)** $$\mathbf{v}(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle$$
$$= \langle 81\cos 57°,\ 81\sin 57° - 32t\rangle$$

**(c)** Find $t$ when $x(t) = 270$.

$$90 + 81t\cos 57° = 270$$
$$81t\cos 57° = 180$$
$$t = \frac{180}{81\cos 57°}$$

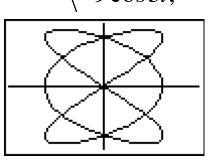
The ball is "over" the player after about 4.08 seconds.

**(d)** Find $y(t)$ when $t = 4.08$.

$$y(4.08) = 81(4.08)\sin 57° - 16(4.08)^2$$
$$\approx 10.821$$

It is unlikely that the player will be able to catch the ball without backing up.

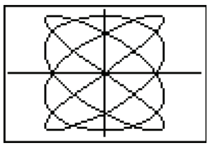
**35.** $$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}$$
$$= \frac{d}{dt}\langle \cos 3t,\ \sin 2t\rangle$$
$$= \langle -3\sin 3t,\ 2\cos 2t\rangle$$
$$\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt}$$
$$= \frac{d}{dt}\langle -3\sin 3t,\ 2\cos 2t\rangle$$
$$= \langle -9\cos 3t,\ -4\sin 2t\rangle$$



$[-1.6, 1.6]$ by $[-1.1, 1.1]$
$0 \le t \le 2\pi$

**36.** $\mathbf{v}(t) = \dfrac{d\mathbf{r}(t)}{dt}$
$= \dfrac{d}{dt}\langle \sin 4t, \cos 3t \rangle$
$= \langle 4\cos 4t, -3\sin 3t \rangle$

$\mathbf{a}(t) = \dfrac{d\mathbf{v}(t)}{dt}$
$= \dfrac{d}{dt}\langle 4\cos 4t, -3\sin 3t \rangle$
$= \langle -16\sin 4t, -9\cos 3t \rangle$

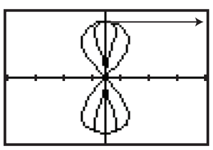


$[-1.6, 1.6]$ by $[-1.1, 1.1]$
$0 \le t \le 2\pi$

**37.** **(a)** $\mathbf{v}(t) = \dfrac{d}{dt}\langle \sin 4t \cos t, \sin 2t \rangle$
$= \langle 4\cos 4t \cos t - \sin t \sin 4t, 2\cos 2t \rangle \Big|_{t=(5\pi/4)}$
$= \langle 2\sqrt{2}, 0 \rangle$

Speed: $2\sqrt{2}$

**(b)**



$[-4, 4]$ by $[-1.2, 1.2]$
$0 \le t \le 2\pi$

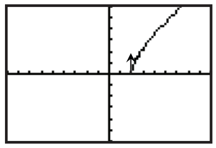
**(c)** To the right

**38.** **(a)** $\mathbf{v}(t) = \dfrac{d}{dt}\left\langle e^t + e^{-t},\ e^t - e^{-t} \right\rangle$
$= \left\langle e^t - e^{-t},\ e^t + e^{-t} \right\rangle$

**(b)** $\displaystyle\lim_{t\to\infty} \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \lim_{t\to\infty} \frac{e^t + e^{-t}}{e^t - e^{-t}} = \lim_{t\to\infty} \frac{e^{2t}+1}{e^{2t}-1} = 1$

**(c)** For any $t$,
$x^2 - y^2 = (e^t + e^{-t})^2 - (e^t - e^{-t})^2$
$= e^{2t} + 2 + e^{-2t} - (e^{2t} - 2 + e^{-2t})$
$= 4$

**(d)** The velocity at $t=0$ is $\langle 0,\ 2\rangle$.



[–9, 9] by [–6, 6]
$0 \le t \le 3$

**39. (a)** $\left\langle \int_0^3 3t^2 - 2t\, dt,\ \int_0^3 1+\cos \pi t\, dt \right\rangle + \langle 2, 6\rangle$

$= \left\langle \left(t^3 - t^2\right)\Big|_0^3,\ \left(t + \dfrac{1}{\pi}\sin \pi t\right)\Big|_0^3 \right\rangle + \langle 2, 6\rangle$

$= \langle 2+18, 6+3\rangle$

$= \langle 20, 9\rangle$

**(b)** $\int_0^3 \sqrt{(3t^2-2t)^2 + (1+\cos \pi t)^2}\, dt$

$\approx 19.343$

**40. (a)** $\left\langle \int_0^3 2\pi \cos 4\pi t dt,\ \int_0^3 4\pi \sin 2\pi t dt \right\rangle + \langle 7, 2\rangle$

$= \left\langle \dfrac{1}{2}\sin 4\pi t\Big|_0^3,\ -2\cos 2\pi t\Big|_0^3 \right\rangle + \langle 7, 2\rangle$

$= \langle 7+0,\ 2+0\rangle$

$= \langle 7,\ 2\rangle$

**(b)** $\int_0^3 \sqrt{(2\pi \cos 4\pi t)^2 + (4\pi \sin 2\pi t)^2}\, dt$

$\approx 28.523$

**41. (a)** $\left\langle \int_0^3 (t+1)^{-1} dt,\ \int_0^3 (t+2)^{-2} dt \right\rangle + \langle 3, -2\rangle$

$= \left\langle \ln(t+1)\Big|_0^3,\ -(t+2)^{-1}\Big|_0^3 \right\rangle + \langle 3, -2\rangle$

$= \langle 3+\ln 4,\ -1.7\rangle$

**(b)** $\int_0^3 \sqrt{((t+1)^{-1})^2 + ((t+2)^{-2})^2}\, dt$

$\approx 1.419$

**42. (a)** $\left\langle \int_0^3 e^t - t\, dt,\ \int_0^3 e^t + t\, dt \right\rangle + \langle 1, 1\rangle$

$= \left\langle \left(e^t - \dfrac{t^2}{2}\right)\Big|_0^3,\ \left(e^t + \dfrac{t^2}{2}\right)\Big|_0^3 \right\rangle + \langle 1, 1\rangle$

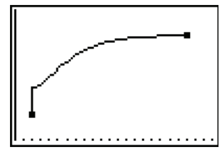
$= \langle 1+14.586, 1+23.586\rangle$

$= \langle 15.586, 24.586\rangle$

**(b)** $\int_0^3 \sqrt{(e^t - t)^2 + (e^t + t)^2}\, dt$

$\approx 27.791$

**43.** The parametric equations are $x = t^3 - t^2 + 2$ and $y = t + \dfrac{1}{\pi}\sin \pi + 6$.

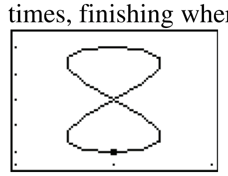


[0, 23] by [5, 10]
$0 \le t \le 3$

**44.** The parametric equations are

$x = \dfrac{1}{2}\sin(4\pi t) + 7$ and $y = -2\cos(2\pi t) + 4$.

(Note: The particle traverses the Figure-8 three times, finishing where it started.)



[6, 8] by [1.5, 6.5]
$0 \le t \le 3$

**45. (a)** $\mathbf{v}(t) = \dfrac{d}{dt}\left\langle 5\cos \dfrac{\pi}{6}t, 3\sin \dfrac{\pi}{6}t \right\rangle$

$\mathbf{v}(t) = \left\langle -\dfrac{5}{6}\pi \sin \dfrac{\pi}{6}t,\ \dfrac{1}{2}\pi \cos \dfrac{\pi}{6}t \right\rangle$

$|\mathbf{v}(t)| = \sqrt{\left(-\dfrac{5}{6}\pi \sin \dfrac{\pi}{6}t\right)^2 + \left(\dfrac{1}{2}\pi \cos \dfrac{\pi}{6}t\right)^2}$

$|\mathbf{v}(2)| = \pi\sqrt{\dfrac{7}{12}} \approx 2.399$

**(b)** $\mathbf{a}(t) = \dfrac{d}{dt}\left\langle -\dfrac{5}{6}\pi \sin \dfrac{\pi}{6}t,\ \dfrac{1}{2}\pi \cos \dfrac{\pi}{6}t \right\rangle$

$\mathbf{a}(2) = \left\langle -\dfrac{5\pi^2}{36}\cos \dfrac{\pi}{6}t, -\dfrac{\pi^2}{12}\sin \dfrac{\pi}{6}t \right\rangle\Big|_{t=2}$

$\mathbf{a}(2) = \left\langle -\dfrac{5\pi^2}{72}, -\dfrac{\pi^2\sqrt{3}}{24} \right\rangle$

**(c)** $x^2 = \left(5\cos\frac{\pi}{6}t\right)^2$

$y^2 = \left(3\sin\frac{\pi}{6}t\right)^2$

$\frac{x^2}{25} + \frac{y^2}{9} = 1$

**46.** **(a)** $\mathbf{v}(t) = \frac{d}{dt}\langle \sec \pi t, \tan \pi t\rangle$

$\mathbf{v}\left(\frac{1}{4}\right) = \left\langle \pi \sec \pi t \tan \pi t, \pi \sec^2 \pi t\right\rangle\Big|_{t=1/4}$

$\mathbf{v}\left(\frac{1}{4}\right) = \left\langle \sqrt{2}\pi,\ 2\pi\right\rangle$

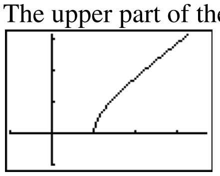
speed: $\sqrt{\left(\sqrt{2}\pi\right)^2 + (2\pi)^2} = \sqrt{6}\pi$

**(b)** $x^2 = (\sec \pi t)^2$

$y^2 = (\tan \pi t)^2$

$x^2 - y^2 = 1$

**(c)** The upper part of the right branch:



$[-1, 3.7]$ by $[-1, 3.1]$
$0 \le t < 1/2$

**47.** **(a)** $\mathbf{v}(t) = \frac{d}{dt}\left\langle \frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right\rangle$

$\mathbf{v}(t) = \left\langle -\frac{4t}{(1+t^2)^2}, \frac{2-2t^2}{(1+t^2)^2}\right\rangle$

**(b)** No; the $x$-component of velocity is zero only if $t = 0$, while the $y$-component of velocity is zero only if $t = 1$. At no time will the velocity be $\langle 0, 0\rangle$.

**(c)** $\lim_{t\to\infty} \left\langle \frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right\rangle = \langle -1,\ 0\rangle$

**48.** **(a)** $\mathbf{v}(t) = \frac{d}{dt}\langle \sin t, \cos 2t\rangle$

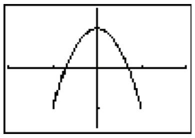
$\mathbf{v}(t) = \langle \cos t, -2\sin 2t\rangle$

**(b)** $\mathbf{v}(t) = \langle 0, 0\rangle$ where $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$.

**(c)** $\cos 2t = 1 - 2\sin^2 t$

$y = 1 - 2x^2$

**(d)**



[−2, 2] by [−1.5, 1.5]
$0 \le t \le 2\pi$

**49. (a)** $\frac{d}{dt}\left\langle e^t \sin t,\ e^t \cos t\right\rangle = \left\langle e^t \sin t + e^t \cos t,\ e^t \cos t - e^t \sin t\right\rangle$

$m = \left.\frac{e^t \cos t - e^t \sin t}{e^t \sin t + e^t \cos t}\right|_{t=\frac{\pi}{2}} = -1$

**(b)** $\left\langle e^1 \sin(1) + e^1 \cos\ (1),\ e^1 \cos(1) - e^1 \sin(1)\right\rangle = \langle 3.756, -\ 0.817\rangle$

$|\mathbf{v}(t)| = \sqrt{(3.756)^2 + (-0.817)^2} \approx 3.844$

**(c)** $\int_0^1 \sqrt{(e^t \sin t + e^t \cos t)^2 + (e^t \cos t - e^t \sin t)^2}\, dt \approx 2.430$

**50. (a)** $\frac{d}{dt}\left\langle t^2 - 3,\ \frac{2}{3}t^3\right\rangle = \langle 2t,\ 2t^2\rangle$

$|\mathbf{v}(4)| = \sqrt{(2(4))^2 + (2(4)^2)^2}$

$|\mathbf{v}(4)| = \sqrt{1088} \approx 32.985$

**(b)** $\int_0^4 ((2t)^2 + (2t^2)^2)^{1/2}\, dt \approx 46.062$

**(c)** $t = \sqrt{3+x}$

$y = \frac{2}{3}(3+x)^{3/2}$

$\frac{dy}{dx} = (3+x)^{1/2}$

**51. (a)** $3 + \int_2^4 (2 + \sin(t^2))\ dt \approx 3.942$

**(b)** $y - y_1 = m(x - x_1)$

$y - 5 = \frac{-6}{2 + \sin 4}(x - 3)$

**(c)** Speed $= \sqrt{(2 + \sin 4)^2 + (-6)^2} \approx 6.127$

**(d)** $\langle 8\cos 16,\ 2(2 + \sin 16) + 7(8)\cos 16\rangle \approx \langle -7.661, -50.205\rangle$

**52.** **(a)** $y - y_1 = m(x - x_1)$

$$y - 5 = \frac{3\cos 4}{\sin 8}(x - 4)$$

**(b)** Speed $= \sqrt{(3\cos 4)^2 + (\sin 8)^2} \approx 2.196$

**(c)** $\int_0^1 \sqrt{(3\cos(t^2))^2 + (\sin(t^3))^2}\, dt \approx 2.741$

**(d)** $\left(4 + \int_2^3 \sin(t^3)\, dt,\ 5 + \int_2^3 3\cos(t^2)\, dt\right)$

$\approx (4.004,\ 5.724)$

**53.** False; for example, $u$ and $-1(u)$ have opposite directions.

**54.** False; for example, $\langle\sqrt{3},\ 0\rangle + \langle 0,\ 1\rangle = \langle\sqrt{3},\ 1\rangle$, which has a direction angle of 30°.

**55.** E; $\frac{d}{dt}\left\langle t^2 + 1,\ \ln(2t+3)\right\rangle = \left\langle 2t,\ \frac{1}{2t+3}\right\rangle$

$$\frac{d}{dt}\left\langle 2t,\ \frac{1}{2t+3}\right\rangle = \left\langle 2,\ -\frac{4}{(2t+3)^2}\right\rangle$$

**56.** D; $\left\langle 4 + \int_0^2 \cos(t^2)\, dt,\ 7 + \int_0^2 \sin(t^3)\, dt\right\rangle$

$\langle 4 + 0.461,\ 7 + 0.452\rangle$
$= \langle 4.461,\ 7.452\rangle$

**57.** B; $x_1 = 7\cos 40 = 5.36$
$y_1 = 7\sin 40 = 4.50$
$x_2 = 4\cos 140 = -3.06$
$y_2 = 4\sin 140 = 2.57$
$\sqrt{(5.36 - 3.06)^2 + (4.50 + 2.57)^2}$
$= 7.435$

**58.** B; $\frac{dx}{dt} = 2\cos 2t$

$$\frac{dy}{dt} = -5\sin 5t$$

Speed $= \sqrt{(2\cos 4)^2 + (-5\sin 10)^2} = 3.018$

**59.** The velocity vector is $\left\langle -x,\ \sqrt{1-x^2}\right\rangle$, which has slope $-\frac{\sqrt{1-x^2}}{x}$. The acceleration vector is

$$\left\langle \frac{d}{dt}(-x),\ \frac{d}{dt}\left(\sqrt{1-x^2}\right)\right\rangle = \left\langle -\frac{dx}{dt},\ \frac{-x}{\sqrt{1-x^2}}\frac{dx}{dt}\right\rangle$$

$= \left\langle x,\ \frac{x^2}{\sqrt{1-x^2}}\right\rangle$, which has slope $\frac{x}{\sqrt{1-x^2}}$.

Since the slopes are negative reciprocals of each other, the vectors are orthogonal.

**60.** The position vector is $\langle \cos t,\ \sin t\rangle$, which has slope tan $t$. The velocity vector is $\langle -\sin t,\ \cos t\rangle$, which has slope $-\frac{1}{\tan t}$. The acceleration vector is $\langle -\cos t,\ -\sin t\rangle$, which has slope tan $t$. The velocity slope is the negative reciprocal of the position and acceleration slopes, so velocity is orthogonal to position and to acceleration.

**61.** **(a)** $t - 3 = \frac{3t}{2} - 4$

$$t - \frac{3t}{2} = -1$$
$$t = 2$$

Since $t = 2$ also solves $(t-3)^2 = \frac{3t}{2} - 2$, the particles collide when $t = 2$.

**(b)** First particle: $\mathbf{v}_1(2) = \langle 1,\ -2\rangle$, so the direction unit vector is $\left\langle \frac{1}{\sqrt{5}},\ -\frac{2}{\sqrt{5}}\right\rangle$.

Second particle: $\mathbf{v}_2(t) = \left\langle \frac{3}{2},\ \frac{3}{2}\right\rangle$, so the direction unit vector is $\left\langle \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right\rangle$.

**62.** **(a)** Referring to the figure, look at the circular arc from the point where $t = 0$ to the point "$m$". On the one hand, this arc has length given by $r_0\theta$, but it also has length given by $vt$. Setting these two quantities equal gives the result.

**(b)** $\mathbf{v}(t) = \left\langle -v\sin\frac{vt}{r_0},\ v\cos\frac{vt}{r_0}\right\rangle$ and

$$\mathbf{a}(t) = \left\langle -\frac{v^2}{r_0}\cos\frac{vt}{r_0},\ -\frac{v^2}{r_0}\sin\frac{vt}{r_0}\right\rangle$$
$$= -\frac{v^2}{r_0}\left\langle \cos\frac{v^2}{r_0},\ \sin\frac{vt}{r_0}\right\rangle$$

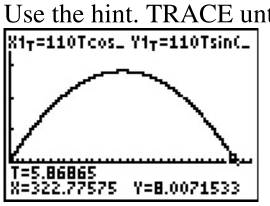
**(c)** From part (b), $\mathbf{a}(t) = -\left(\frac{v}{r_0}\right)^2 \mathbf{r}(t)$. So, by Newton's second law, $\mathbf{F} = -m\left(\frac{v}{r_0}\right)^2 \mathbf{r}$. Substituting for **F** in the law of gravitation gives the result.

**(d)** Set $\frac{vT}{r_0} = 2\pi$ and solve for $vT$.

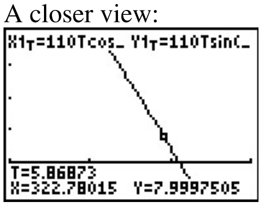
**(e)** Substitute $\frac{2\pi r_0}{T}$ for $v$ in $v^2 = \frac{GM}{r_0}$ and solve for $T^2$.

$$\left(\frac{2\pi r_0}{T}\right)^2 = \frac{GM}{r_0}$$
$$\frac{4\pi^2 r_0{}^2}{T^2} = \frac{GM}{r_0}$$
$$\frac{1}{T^2} = \frac{GM}{4\pi^2 r_0{}^3}$$
$$T^2 = \frac{4\pi^2}{GM} r_0{}^3$$

**63.** Use the hint. TRACE until $Y$ is approximately 8.




A closer view:




The screens confirm that the ball is at a height of 8 feet after about 5.069 seconds when it is about 323 feet from home plate.

**64.** Let $\mathbf{u} = \langle a, b \rangle$ be one of the vectors. It has slope $\frac{b}{a}$, so the perpendicular vector **v** must have slope $-\frac{a}{b}$.

Thus $\mathbf{v} = \langle kb, -ka \rangle$ for some nonzero scalar $k$, and the dot product is

$u \cdot \mathbf{v} = \langle a, b \rangle, \ \langle kb, -ka \rangle = kab + (-kab) = 0.$

**65.** **(a)** The diagram shows, by vector addition, that **v** + **w** = **u**, so **w** = **u** – **v**.

**(b)** This is just the Law of Cosines applied to the triangle, the sides of which are the magnitudes of the vectors.

**(c)** By the HMT Rule, $w = \langle u_1 - v_1,\ u_2 - v_2 \rangle$. So

$$
\begin{aligned}
|\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2 &= \left(u_1^2 + u_2^2\right) + \left(v_1^2 + v_2^2\right) - \left[\left(u_1 - v_1\right)^2 + \left(u_2 - v_2\right)^2\right] \\
&= u_1^2 + u_2^2 + v_1^2 + v_2^2 - \left[u_1^2 - 2u_1v_1 + v_1^2 + u_2^2 - 2u_2v_2 + v_2^2\right] \\
&= 2u_1v_1 + 2u_2v_2 \\
&= 2\left(u_1v_1 + u_2v_2\right)
\end{aligned}
$$

**(d)** From part (b), $|\mathbf{w}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta$, so $|\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2 = 2|\mathbf{u}||\mathbf{v}|\cos\theta$.

From part (c), $|\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2 = 2\left(u_1v_1 + u_2v_2\right)$. Substituting, we get $2\left(u_1v_1 + u_2v_2\right) = 2|\mathbf{u}||\mathbf{v}|\cos\theta$, so $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 = |\mathbf{u}||\mathbf{v}|\cos\theta$.

**Section 11.3** Polar Functions (pp. 555–566)

**Quick Review 11.3**

**1.** $x = 4\ \cos 30 = 2\sqrt{3}$
$y = 4\ \sin 30 = 2$
$\left\langle 2\sqrt{3},\ 2\right\rangle$

**2.** $A = \pi r^2 \dfrac{30}{360} = \pi(6)^2 \dfrac{30}{360} = 3\pi$

**3.** $A = \pi r^2\ \dfrac{\pi}{8}\left(\dfrac{1}{2\pi}\right) = \dfrac{1}{16}\pi\ (8)^2 = 4\pi$

**4.** $x^2 + y^2 = 25$

**5.** Graph $y = \left(\dfrac{4 - x^2}{3}\right)^{1/2}$ and $y = -\left(\dfrac{4 - x^2}{3}\right)^{1/2}$.

**6.**
$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\
&= \frac{\frac{d}{dt}(5\sin t)}{\frac{d}{dt}(3\cos t)} \\
&= \frac{5\cos t}{-3\sin t} \\
&= -\frac{5}{3}\cot t
\end{aligned}
$$

**7.** $-\dfrac{5}{3}\cot(2) = 0.763$

**8.** $-\frac{5}{3}\cot t = 0, \quad \text{so} \quad t = \frac{\pi}{2} \quad \text{or} \quad \frac{3\pi}{2}$

$x = 3\cos\frac{\pi}{2} = 0 \quad x = 3\cos\frac{3\pi}{2} = 0$

$y = 5\sin\frac{\pi}{2} = 5 \quad y = 5\sin\frac{3\pi}{2} = -5$

(0, 5) and (0, −5)

**9.** $-3\sin t = 0$, s $t = 0$ or $t = \pi$.

$x = 3\cos 0 = 3 \quad x = 3\cos\pi = -3$

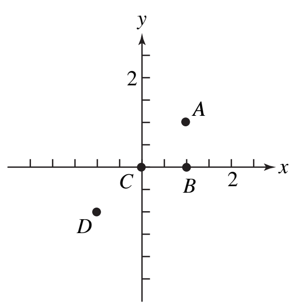
$y = 5\sin 0 = 0 \quad y = 5\sin\pi = 0$

(3, 0) and (−3, 0)

**10.** $\int_0^{\pi} \sqrt{(3\cos t)^2 + (5\sin t)^2}\, dt \approx 12.763$

**Section 11.3** Exercises

**1.**




**(a)** $x = \sqrt{2}\cos\left(\frac{\pi}{4}\right) = 1$

$y = \sqrt{2}\sin\left(\frac{\pi}{4}\right) = 1$

(1, 1)

**(b)** $x = 1\cos(0) = 1$

$y = 1\sin(0) = 0$

(1, 0)

**(c)** $x = 0\cos\left(\frac{\pi}{2}\right) = 0$

$y = 0\sin\left(\frac{\pi}{2}\right) = 0$

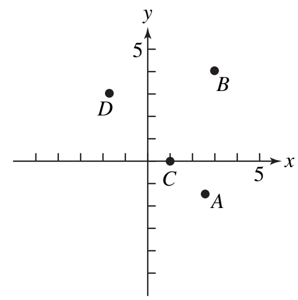
(0, 0)

**(d)** $x = -\sqrt{2}\cos\left(\frac{\pi}{4}\right) = -1$

$y = -\sqrt{2}\sin\left(\frac{\pi}{4}\right) = -1$

(−1, −1)

**2.**




**(a)** $x = -3\cos\left(\frac{5\pi}{6}\right) = \frac{3\sqrt{3}}{2}$

$y = -3\sin\left(\frac{5\pi}{6}\right) = \frac{-3}{2}$

$\left(\frac{3\sqrt{3}}{2}, \frac{-3}{2}\right)$

**(b)** $x = 5\cos\left(\tan^{-1}\left(\frac{4}{3}\right)\right) = 3$

$y = 5\sin\left(\tan^{-1}\left(\frac{4}{3}\right)\right) = 4$

(3, 4)

**(c)** $x = -1\cos 7\pi = 1$

$y = -1\sin 7\pi = 0$

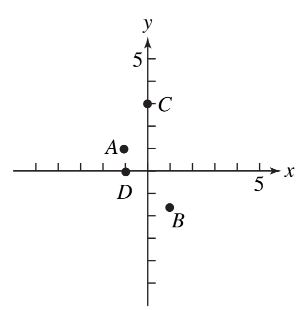
(1, 0)

**(d)** $x = 2\sqrt{3}\cos\left(\frac{2\pi}{3}\right) = -\sqrt{3}$

$y = 2\sqrt{3}\sin\left(\frac{2\pi}{3}\right) = 3$

$\left(-\sqrt{3}, 3\right)$

**3.**




**(a)** $r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1}{-1}\right) = -\frac{5\pi}{4}, \frac{3\pi}{4}$

$\left(\sqrt{2}, \frac{-5\pi}{4}\right)$ and $\left(\sqrt{2}, \frac{3\pi}{4}\right)$

**(b)** $r = \sqrt{1 + \left(-\sqrt{3}\right)^2} = \pm 2$

$\theta = \tan^{-1}\left(\dfrac{-\sqrt{3}}{1}\right) = -\dfrac{\pi}{3}, \dfrac{2\pi}{3}$

$\left(-2, \dfrac{2\pi}{3}\right)$ and $\left(2, -\dfrac{\pi}{3}\right)$

**(c)** $r = \sqrt{0^2 + 3^2} = \pm 3$

$\theta = \tan^{-1}\left(\dfrac{3}{0}\right) = \dfrac{\pi}{2}, \dfrac{5\pi}{2}$

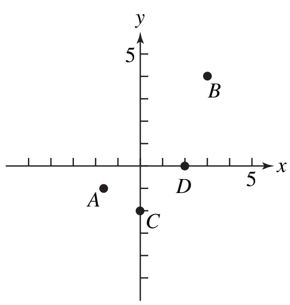
$\left(3, \dfrac{\pi}{2}\right)$ and $\left(3, \dfrac{5\pi}{2}\right)$

**(d)** $r = \sqrt{(-1)^2 + 0^2} = \pm 1$

$\theta = \tan^{-1}\left(\dfrac{0}{1}\right) = 0, \pi$

$(-1, 0)$ and $(1, \pi)$

**4.**




**(a)** $r = \sqrt{\left(-\sqrt{3}\right)^2 + (-1)^2} = \pm 2$

$\theta = \tan^{-1}\left(\dfrac{-1}{-\sqrt{3}}\right) = \dfrac{\pi}{6}, \dfrac{7\pi}{6}$

$\left(-2, \dfrac{\pi}{6}\right)$ and $\left(2, \dfrac{7\pi}{6}\right)$

**(b)** $r = \sqrt{3^2 + 4^2} = \pm 5$

$\theta = \tan^{-1}\left(\dfrac{4}{3}\right)$

$\left(-5, \pi + \tan^{-1}\left(\dfrac{4}{3}\right)\right)$ and $\left(5, \tan^{-1}\left(\dfrac{4}{3}\right)\right)$

**(c)** $r = \sqrt{0^2 + (-2)^2} = \pm 2$

$\theta = \tan^{-1}\left(\dfrac{-2}{0}\right)$

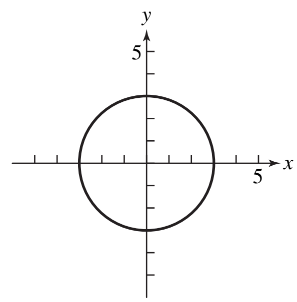
$= -\dfrac{\pi}{2}, \dfrac{3\pi}{2}, \left(2, -\dfrac{\pi}{2}\right)$ and $\left(2, \dfrac{3\pi}{2}\right)$

**(d)** $r = \sqrt{2^2 + 0^2} = \pm 2$

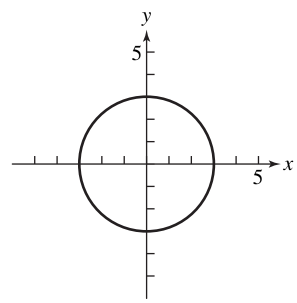
$\theta = \tan^{-1}\left(\dfrac{0}{2}\right) = 0, 2\pi$

$(2, 0)$ and $(2, 2\pi)$

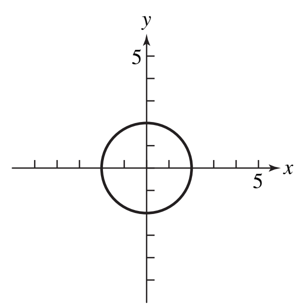
**5.**



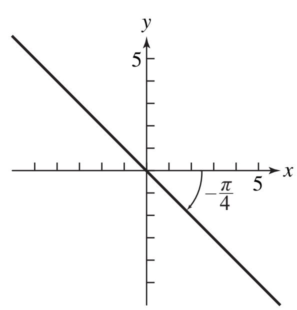

**6.**



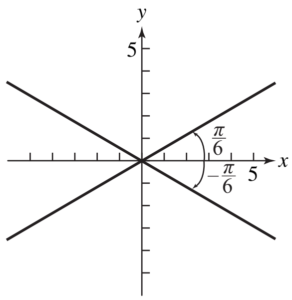

**7.**



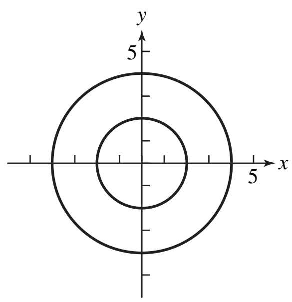

**8.**



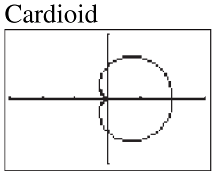
**9.**




**10.**

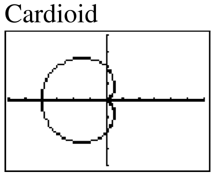



**11.** Cardioid



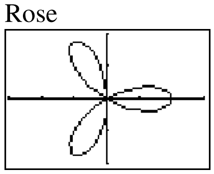
[–3, 3] by [–2, 2]
$0 \le \theta \le 2\pi$

**12.** Cardioid



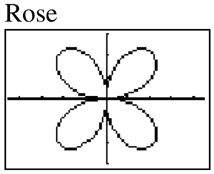
[–6, 6] by [–4, 4]
$0 \le \theta \le 2\pi$

**13.** Rose



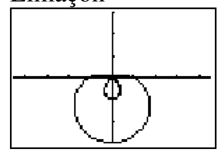
[–3, 3] by [–2, 2]
$0 \le \theta \le \pi$

**14.** Rose



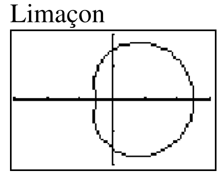
[–4.5, 4.5] by [–3, 3]
$0 \le \theta \le 2\pi$

**15.** Limaçon



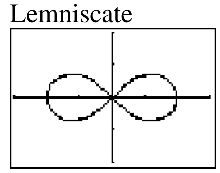
[–4.5, 4.5] by [–3, 3]
$0 \le \theta \le 2\pi$

**16.** Limaçon



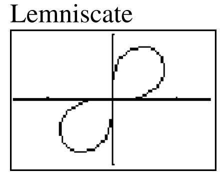
[–3, 3] by [–2, 2]
$0 \le \theta \le 2\pi$

**17.** Lemniscate



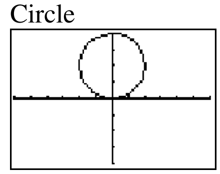
[–3, 3] by [–2, 2]
$-\pi/4 \le \theta \le \pi/4$

**18.** Lemniscate



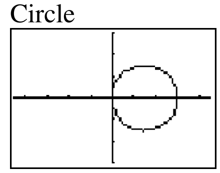
[–1.5, 1.5] by [–1, 1]
$0 \le \theta \le \pi/2$

**19.** Circle



[–6, 6] by [–4, 4]
$0 \le \theta \le \pi$

**20.** Circle



[–4.5, 4.5] by [–3, 3]
$0 \le \theta \le \pi$

**21.** $r = 4\csc\theta$

$1 = \dfrac{4}{r\sin\theta}$

$y = 4$, a horizontal line

**22.** $r = -3\sec\theta$

$1 = \dfrac{-3}{r\cos\theta}$

$x = -3$, a vertical line

**23.** $x + y = 1$, a line

(slope = −1, $y$-intercept = 1)

**24.** $r^2 = x^2 + y^2 = 1$,

a circle (center = (0, 0), radius = 1)

**25.** $r = \dfrac{5}{\sin\theta - 2\cos\theta}$

$1 = \dfrac{5}{r\sin\theta - 2r\cos\theta}$

$y - 2x = 5$, a line (slope = 2, $y$-intercept = 5)

**26.**
$$r^2 \sin 2\theta = 2$$
$$r^2(2\sin\theta\cos\theta) = 2$$
$$2xy = 2$$
$$xy = 1, \text{ a hyperbola}$$

**27.** $r^2\cos^2\theta = r^2\sin\theta$

$x^2 = y^2$, the union of two lines: $y = \pm x$

**28.**
$$r^2 = -4r\cos\theta$$
$$x^2 + y^2 = -4x$$
$$x^2 + 4x + 4 - 4 + y^2 = 0$$
$$(x+2)^2 + y^2 = 4, \text{ a circle}$$

(center = (−2, 0), radius = 2)

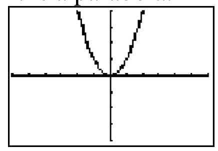
**29.**
$$r^2 = 8r\sin\theta$$
$$x^2 + y^2 - 8r\sin\theta + 16 - 16 = 0$$
$$x^2 + (y-4)^2 = 16, \text{ a circle}$$

(center = (0, 4), radius = 4)

**30.**
$$r = 2\cos\theta + 2\sin\theta$$
$$r^2 = 2r\cos\theta + 2r\sin\theta$$
$$x^2 + y^2 = 2x + 2y$$
$$(x-1)^2 + (y-1)^2 = 2, \text{ a circle}$$

$\left(\text{center} = (1, 1), \text{ radius} = \sqrt{2}\right)$

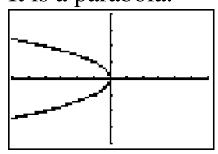
**31.** It is a parabola.



[−6, 6] by [−4, 4]

$0 \le \theta \le 2\pi$

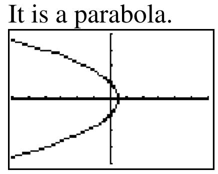
**32.** It is a parabola.



[−6, 6] by [−4, 4]

$0 \le \theta \le 2\pi$

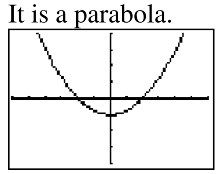
**33.** It is a parabola.



[−6, 6] by [−4, 4]

$0 \le \theta \le 2\pi$

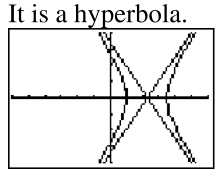
**34.** It is a parabola.



[−6, 6] by [−4, 4]

$0 \le \theta \le 2\pi$

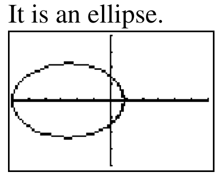
**35.** It is a hyperbola.



[−6, 6] by [−4, 4]

$0 \le \theta \le 2\pi$

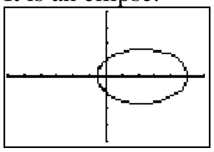
**36.** It is an ellipse.



[−6, 6] by [−4, 4]

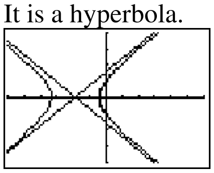
$0 \le \theta \le 2\pi$

**37.** It is an ellipse.



[−6, 6] by [−4, 4]
$0 \le \theta \le 2\pi$

**38.** It is a hyperbola.



[−6, 6] by [−4, 4]
$0 \le \theta \le 2\pi$

**39.** $r = -1 + \sin\theta$

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}((-1+\sin\theta)\sin\theta)}{\frac{d}{d\theta}((-1+\sin\theta)\cos\theta)}$$

$$\frac{dy}{dx} = \frac{(2\sin\theta - 1)\cos\theta}{\cos^2\theta - \sin^2\theta - \sin\theta}$$

At $\theta = 0$: $-1$

At $\theta = \pi$: 1

**40.** $r = \cos 2\theta$

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}(\cos 2\theta \sin\theta)}{\frac{d}{d\theta}(\cos 2\theta \cos\theta)}$$

$$\frac{dy}{dx} = \frac{\cos\theta\cos 2\theta - 2\sin\theta\sin 2\theta}{-\sin\theta\cos 2\theta - 2\cos\theta\sin 2\theta}$$

*At* $\theta = 0$ : undefined

At $\theta = -\frac{\pi}{2}$: 0

At $\theta = \frac{\pi}{2}$: 0

At $\theta = \pi$: undefined

**41.** $r = 2 - 3\sin\theta$

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}(2-3\sin\theta)\sin\theta}{\frac{d}{d\theta}(2-3\sin\theta)\cos\theta}$$

$$\frac{dy}{dx} = \frac{(2-6\sin\theta)\cos\theta}{\sin\theta\,(3\sin\theta - 2) - 3\cos^2\theta}$$

At (2, 0): $-\frac{2}{3}$

At $\left(-1, \frac{\pi}{2}\right)$: 0

At $(2, \pi)$: $\frac{2}{3}$

At $\left(5, \frac{3\pi}{2}\right)$: 0

**42.** $r = 3\,(1 - \cos\theta)$

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}\left(3\,(1-\cos\theta)\sin\theta\right)}{\frac{d}{d\theta}\left(3\,(1-\cos\theta)\cos\theta\right)}$$

$$\frac{dy}{dx} = \frac{-6\cos^2\theta + 3\cos\theta + 3}{3\sin\theta\,(2\cos\theta - 1)}$$

At $\left(1.5, \frac{\pi}{3}\right)$: undefined

At $\left(4.5, \frac{2\pi}{3}\right)$: 0

At $(6, \pi)$: undefined

At $\left(3, \frac{3\pi}{2}\right)$: 1

**43.** $\int_0^{2\pi} \frac{1}{2}(4 + 2\cos\theta)^2\, d\theta$

$$= \int_0^{2\pi} \frac{1}{2}(16 + 16\cos\theta + 4\cos^2\theta)\, d\theta$$

$$= \int_0^{2\pi} (8 + 8\cos\theta + 1 + \cos 2\theta)\, d\theta$$

$$= \left[9\theta + 8\sin\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi}$$

$$= 18\pi$$

**44.** $\int_0^{2\pi} \frac{1}{2}(2 + 2\sin\theta)^2\, d\theta$

$$= \int_0^{2\pi} \frac{1}{2}(4 + 8\sin\theta + 4\sin^2\theta)\, d\theta$$

$$= \int_0^{2\pi} (2 + 4\sin\theta + 1 - \cos 2\theta)\, d\theta$$

$$= \left[3\theta - 4\cos\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi}$$

$$= 6\pi - 4 + 4$$

$$= 6\pi$$

**45.** $\int_{-\pi/4}^{\pi/4}\frac{1}{2}(\cos 2\theta)^2\,d\theta = \int_0^{\pi/4}\cos^2(2\theta)\,d\theta$

$= \int_0^{\pi/4}\left(\frac{1}{2}+\frac{\cos 4\theta}{2}\right)d\theta$

$= \left[\frac{\theta}{2}+\frac{\sin 4\theta}{8}\right]_0^{\pi/4}$

$= \frac{\pi}{8}$

**46.** $\int_0^{2\pi}\frac{1}{2}(2\sin 4\theta)^2\,d\theta = \int_0^{2\pi}2\sin^2(4\theta)\,d\theta$

$= \int_0^{2\pi}(1-\cos 8\theta)\,d\theta$

$= \left[\theta-\frac{\sin 8\theta}{8}\right]_0^{2\pi}$

$= 2\pi$

**47.** $\int_{-\pi/4}^{\pi/4}\frac{1}{2}(4\cos 2\theta)\,d\theta = \sin 2\theta\Big|_{-\pi/4}^{\pi/4}$

$= 2$

**48.** $6\int_0^{\pi/3}\frac{1}{2}(2\sin 3\theta)\,d\theta = -2\cos 3\theta\Big|_0^{\pi/3}$

$= 2-(-2) = 4$

**49.** $\int_0^{2\pi}\frac{1}{2}(3-2\cos\theta)^2\,d\theta$

$= \int_0^{2\pi}\frac{1}{2}(9-12\cos\theta+4\cos^2\theta)\,d\theta$

$= \int_0^{2\pi}\left(\frac{9}{2}-6\cos\theta+1+\cos 2\theta\right)d\theta$

$= \left[\frac{11\theta}{2}-6\sin\theta+\frac{1}{2}\sin 2\theta\right]_0^{2\pi}$

$= 11\pi - 0 = 11\pi$

**50.** $\int_{\pi/6}^{5\pi/6}\frac{1}{2}(2\sin\theta-1)^2\,d\theta$

$= \int_{\pi/6}^{5\pi/6}\left(2\sin^2\theta-2\sin\theta+\frac{1}{2}\right)d\theta$

$= \int_{\pi/6}^{5\pi/6}\left(1-\cos 2\theta-2\sin\theta+\frac{1}{2}\right)d\theta$

$= \left[\frac{3\theta}{2}-\frac{\sin 2\theta}{2}+2\cos\theta\right]_{\pi/6}^{5\pi/6}$

$= \pi-\frac{3\sqrt{3}}{2} = 0.544$

**51.** $2\int_0^{\pi/4}\frac{1}{2}(2\sin\theta)^2 d\theta = \int_0^{\pi/4}4\sin^2\theta d\theta$

$= \int_0^{\pi/4}2(1-\cos 2\theta)\,d\theta$

$= [2\theta-\sin 2\theta]_0^{\pi/4}$

$= \frac{\pi}{2}-1$

**52.** $2\left(\int_0^{\pi/6}\frac{1}{2}(2\sin\theta)^2 d\theta+\int_{\pi/6}^{\pi/2}\frac{1}{2}(1)^2 d\theta\right)$

$= \int_0^{\pi/6}4\sin^2\theta d\theta+\int_{\pi/6}^{\pi/2}1 d\theta$

$= \int_0^{\pi/6}(2-2\cos 2\theta)\,d\theta+\frac{\pi}{3}$

$= [2\theta-\sin 2\theta]_0^{\pi/6}+\frac{\pi}{3}$

$= \frac{2\pi}{3}-\frac{\sqrt{3}}{2}$

**53.** $2\left(\int_0^{\pi/2}\frac{1}{2}[2(1-\cos\theta)]^2 d\theta+\int_{\pi/2}^{\pi}\frac{1}{2}(2)^2 d\theta\right)$

$= \int_0^{\pi/2}4(1-2\cos\theta+\cos^2\theta)d\theta+\int_{\pi/2}^{\pi}4d\theta$

$= \int_0^{\pi/2}(4-8\cos\theta+2(1+\cos 2\theta))\,d\theta+2\pi$

$= [6\theta-8\sin\theta+\sin 2\theta]_0^{\pi/2}+2\pi$

$= 5\pi-8$

**54.** $4\int_0^{\pi/2}\frac{1}{2}(2(1-\cos\theta))^2 d\theta$

$= 4\int_0^{\pi/2}(2-4\cos\theta+2\cos^2\theta)\,d\theta$

$= \int_0^{\pi/2}(8-16\cos\theta+4(1+\cos 2\theta))\,d\theta$

$= [12\theta-16\sin\theta+2\sin 2\theta]_0^{\pi/2}$

$= 6\pi-16$

**55.** The requested area is inside of the upper semicircle and outside of the portion of the cardioid that is in Quadrants I and II.

$2\pi-\int_0^{\pi}\frac{1}{2}(2(1-\sin\theta))^2 d\theta$

$= 2\pi-\int_0^{\pi}(2-4\sin\theta+2\sin^2\theta)d\theta$

$= 2\pi-\int_0^{\pi}(2-4\sin\theta+1-\cos 2\theta)d\theta$

$= 2\pi-\left[3\theta+4\cos\theta-\frac{1}{2}\sin 2\theta\right]_0^{\pi}$

$= 8-\pi$

**56.** $4\int_{-\pi/6}^{\pi/6}\frac{1}{2}((4\cos 2\theta)^2-2^2)\,d\theta$

$=4\int_0^{\pi/6}(16\cos^2 2\theta-4)\,d\theta$

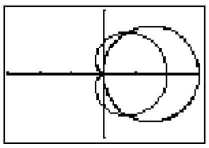
$=4\int_0^{\pi/6}(8(1+\cos 4\theta)-4)\,d\theta$

$=\int_0^{\pi/6}(16+32\cos 4\theta)\,d\theta$

$=[16\theta+8\sin 4\theta]_0^{\pi/6}$

$=\frac{8\pi}{3}+4\sqrt{3}$

**57.**



[–3, 3] by [–2, 2]

$0\le\theta\le\pi$ for the circle
$0\le\theta\le 2\pi$ for the cardioid

$2\int_0^{\pi/3}\frac{1}{2}((3\cos\theta)^2-(1+\cos\theta)^2)d\theta$

$=\int_0^{\pi/3}(8\cos^2\theta-1-2\cos\theta)\,d\theta$

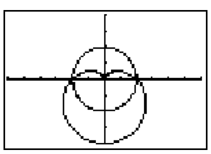
$=\int_0^{\pi/3}(4(1+\cos 2\theta)-1-2\cos\theta)d\theta$

$=[3\theta+2\sin 2\theta-2\sin\theta]_0^{\pi/3}$

$=\pi+\sqrt{3}-\sqrt{3}-0$

$=\pi$

**58.**



[–6, 6] by [–4, 4]

$0\le\theta\le 2\pi$

$\int_0^{\pi}\frac{1}{2}(2^2-(2(1-\sin\theta))^2)\,d\theta$

$=\int_0^{\pi}(2-2+4\sin\theta-2\sin^2\theta)\,d\theta$

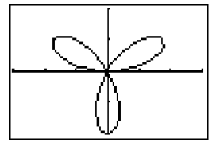
$=\int_0^{\pi}(4\sin\theta-1+\cos 2\theta)\,d\theta$

$=\left[-4\cos\theta-\theta+\frac{1}{2}\sin 2\theta\right]_0^{\pi}$

$=4-\pi-(-4)$

$=8-\pi$

**59.**



[–3, 3] by [–2, 2]

$0\le\theta\le\pi$

$\int_0^{\pi}\frac{1}{2}(2\sin 3\theta)^2\,d\theta=\int_0^{\pi}2\sin^2 3\theta d\theta$

$=\int_0^{\pi}(1-\cos 6\theta)d\theta$

$=\left[\theta-\frac{1}{6}\sin 6\theta\right]_0^{\pi}$

$=\pi$

$$\text{Slope}=\left.\frac{\frac{d}{d\theta}(2\sin 3\theta\sin\theta)}{\frac{d}{d\theta}(2\sin 3\theta\cos\theta)}\right|_{\theta=\pi/4}$$

$$=\left[\frac{6\sin\theta\cos 3\theta+2\cos\theta\sin 3\theta}{6\cos\theta\cos 3\theta-2\sin\theta\sin 3\theta}\right]_{\theta=\pi/4}$$

$$=\frac{1}{2}$$

**60.** **(a)** $\int_0^{3/4}\left(\sqrt{1+y^2}-\frac{5y}{3}\right]dy$

$$=\left[\frac{\ln\left(\left|\sqrt{y^2+1}+y\right|\right)}{2}+\frac{y\sqrt{y^2+1}}{2}-\frac{5y^2}{6}\right]_0^{3/4}$$

$=0.347$

**(b)** $x=r\cos\theta$
$y=r\sin\theta$

$$x^2=1+y^2$$
$$x^2-y^2=1$$
$$r^2(\cos^2\theta-\sin^2\theta)=1$$
$$r^2=\frac{1}{\cos^2\theta-\sin^2\theta}$$

**(c)** Let $\alpha=\tan^{-1}\left(\frac{3}{5}\right)$.

Then the area is

$\int_0^{\alpha}\frac{1}{2}\left(\frac{1}{\cos^2\theta-\sin^2\theta}\right)d\theta.$

**61.** True; polar coordinates determine a unique point.

**62.** False; integrating from 0 to $2\pi$ traverses the curve twice, giving twice the area. The correct upper limit of integration is $\pi$.

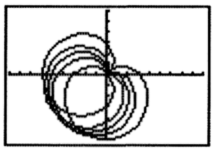
**63.** D

**64.** E

**65.** B

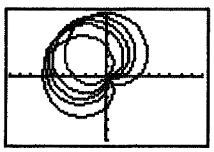
**66.** D

**67.** **(a)**



[–9, 9] by [–6, 6]

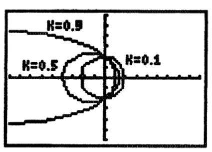
**(b)**



[–9, 9] by [–6, 6]

**(c)** The graph of $r_2$ is the graph of $r_1$ rotated counterclockwise about the origin by angle $\alpha$.

**68.** **(a)**

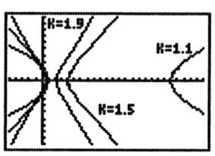



[–9, 9] by [–6, 6]

The graphs are ellipses.

**(b)** As $k \to 0^+$, the graph approaches the circle of radius 2 centered at the origin.

**69.** **(a)**

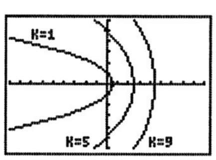



[–5, 25] by [–10, 10]

The graphs are hyperbolas.

**(b)** As $k \to 1^+$, the right branch of the hyperbola goes to infinity and “disappears.” The left branch approaches the parabola $y^2 = 4 - 4x$.

**70.** **(a)**




[–9, 9] by [–6, 6]

The graphs are parabolas.

**(b)** As $k \to 0^+$, the limit of the graph is the negative $x$-axis.

**71.** $d=[(x_2-x_1)^2+(y_2-y_1)^2]^{1/2}$

$d=[(r_2\cos\theta_2-r_1\cos\theta_1)^2+(r_2\sin\theta_2-r_1\sin\theta_1)^2]^{1/2}$

$d=[r_2{}^2\cos^2\theta_2+r_1{}^2\cos^2\theta_1+r_2{}^2\sin^2\theta_2+r_1{}^2\sin^2\theta_1+2\,r_1r_2\cos\theta_1\cos\theta_2+2\,r_1r_2\sin\theta_1\sin\theta_2]^{1/2}$

$d=[r_1{}^2+r_2{}^2-2\,r_1r_2\cos(\theta_1-\theta_2)]^{1/2}$

**72.** **(a)** $\dfrac{1}{2\pi-0}\displaystyle\int_0^{2\pi}a(1-\cos\theta)\,d\theta=\frac{a}{2\pi}[\theta-\sin\theta]_0^{2\pi}=a$

**(b)** $\dfrac{1}{2\pi-0}\displaystyle\int_0^{2\pi}a\,d\theta=\frac{a}{2\pi}[\theta]_0^{2\pi}=a$

**(c)** $\dfrac{1}{\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)}\displaystyle\int_{-\pi/2}^{\pi/2}a\cos\theta\,d\theta=\frac{a}{\pi}\left[\sin\theta\right]_{-\pi/2}^{\pi/2}=\frac{2a}{\pi}$

**73.**
$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2&=(f'(\theta)\cos\theta-f(\theta)\sin\theta)^2+(f'(\theta)\sin\theta+f(\theta)\cos\theta)^2\\&=(f'(\theta)\cos\theta)^2+(f(\theta)\sin\theta)^2+(f'(\theta)\sin\theta)^2+(f(\theta)\cos\theta)^2\\&=(f(\theta))^2(\cos^2\theta+\sin^2\theta)+(f'(\theta))^2(\cos^2\theta+\sin^2\theta)\\&=(f(\theta))^2+(f'(\theta))^2\\&=r^2+\left(\frac{dr}{d\theta}\right)^2\end{aligned}$$

**74.**
$$\begin{aligned}\int_0^{2\pi}\sqrt{(1+\cos\theta)^2+(-\sin\theta)^2}\,d\theta&=\int_0^{2\pi}\sqrt{2\cos\theta+2}\,d\theta\\&=2\int_0^{\pi}\sqrt{2\cos\theta+2}\,d\theta\\&=2\int_0^{\pi}2\cos\left(\frac{\theta}{2}\right)d\theta\\&=8\sin\left(\frac{\theta}{2}\right)\Bigg|_0^{\pi}\\&=8\end{aligned}$$

**Quick Quiz Sections 11.1–11.3**

**1.** A.

**2.** C; $\dfrac{dx}{dt}=\dfrac{d}{dt}(t^3-t^2-1)=3t^2-2t$

$3t^2-2t=0$

$t\,(3t-2)=0$

$t=0,\ \dfrac{2}{3}$

**3.** D.

**4.** **(a)** Area $=\dfrac{1}{2}\displaystyle\int_0^{\pi}(\theta+\sin 2\theta)^2\,d\theta=4.382$

**(b)** $-2 = r\cos\theta = (\theta + \sin 2\theta)\cos\theta$
$\Rightarrow \theta = 2.786$

**(c)** The graph is getting closer to the origin as $\theta$ increases from $\frac{\pi}{3}$ to $\frac{2\pi}{3}$.

**(d)** Maximize $r = \theta + \sin 2\theta$ for $0 \le \theta \le \frac{\pi}{2}$.

$$\frac{dr}{d\theta} = 1 + 2\cos 2\theta$$
$$1 + 2\cos 2\theta = 0$$
$$\cos 2\theta = -\frac{1}{2}$$
$$\theta = \frac{\pi}{3}$$

Since $\frac{dr}{d\theta} = 1 + 2\cos 2\theta > 0$ for $0 < \theta < \frac{\pi}{3}$ and $\frac{dr}{d\theta} = 1 + 2\cos 2\theta < 0$ for $\frac{\pi}{3} < \theta < \frac{\pi}{2}$, there is a maximum of $r$ when $\theta = \frac{\pi}{3}$ by the First Derivative test. The curve is farthest from the origin when $\theta = \frac{\pi}{3}$.

**Chapter 11 Review Exercises** (pp. 567–568)

**1.** **(a)** $3\langle -3, 4\rangle - 4\langle 2, -5\rangle = \langle -17, 32\rangle$

**(b)** $\sqrt{(-17)^2 + (32)^2} = \sqrt{1313}$

**2.** **(a)** $\langle -3, 4\rangle + \langle 2, -5\rangle = \langle -1, -1\rangle$

**(b)** $\sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

**3.** **(a)** $-2\langle -3, 4\rangle = \langle 6, -8\rangle$

**(b)** $\sqrt{6^2 + (-8)^2} = 10$

**4.** **(a)** $5\langle 2, -5\rangle = \langle 10, -25\rangle$

**(b)** $\sqrt{10^2 + (-25)^2} = \sqrt{725} = 5\sqrt{29}$

**5.** $y = \sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$

$x = \cos\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$\left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\rangle$

**6.** $y = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$x = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$\left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle$

**7.** $r = \sqrt{(4)^2 + (-1)^2} = \sqrt{17}$

$2\left\langle \frac{4}{\sqrt{17}}, \frac{-1}{\sqrt{17}}\right\rangle = \left\langle \frac{8}{\sqrt{17}}, \frac{-2}{\sqrt{17}}\right\rangle$

**8.** $-5\left\langle \frac{3}{5}, \frac{4}{5}\right\rangle = \langle -3, -4\rangle$

**9.** **(a)** $y = \frac{\sqrt{3}}{2}x + \frac{1}{4}$

**(b)** $\frac{dy}{dx} = \frac{\frac{d}{dt}\left(\frac{1}{2}\sec t\right)}{\frac{d}{dt}\left(\frac{1}{2}\tan t\right)}$

$\frac{dy}{dx} = \sin t$

$\frac{d^2y}{dx^2} = \left.\frac{\frac{d}{dt}(\sin t)}{\frac{1}{2}\sec^2 t}\right|_{t=\frac{\pi}{3}}$

$\frac{d^2y}{dx^2} = \frac{1}{4}$

**10.** **(a)** $y = -3x + \frac{13}{4}$

**(b)** $\dfrac{dy}{dx} = \dfrac{\frac{d}{dt}\left(1-\frac{3}{t}\right)}{\frac{d}{dt}\left(1+\frac{1}{t^2}\right)}$

$\dfrac{dy}{dx} = \dfrac{\frac{3}{t^2}}{-\frac{2}{t^3}}$

$\dfrac{dy}{dx} = -\dfrac{3}{2}t$

$\dfrac{d^2y}{dx^2} = \left.\dfrac{\frac{d}{dt}\left(-\frac{3}{2}t\right)}{-\frac{2}{t^3}}\right|_{t=2} = \left.\dfrac{3}{4}t^3\right|_{t=2}$

$\dfrac{d^2y}{dx^2} = 6$

**11. (a)** $\dfrac{dy}{dt} = \dfrac{d}{dt}\left(\dfrac{1}{2}\sec t\right) = \dfrac{1}{2}\sec t\,\tan t$

Horizontal: $\dfrac{dy}{dt} = 0 \Rightarrow \tan t = 0 \Rightarrow x = 0$

Also, $\tan t = 0 \Rightarrow \sec t = \pm 1 \Rightarrow y = \pm\dfrac{1}{2}$

The point are $\left(0, -\dfrac{1}{2}\right)$ and $\left(0, \dfrac{1}{2}\right)$.

**(b)** $\dfrac{dx}{dt} = \dfrac{d}{dt}\left(\dfrac{1}{2}\tan t\right) = \dfrac{1}{2}\sec^2 t$

Vertical: $\dfrac{dx}{dt} = 0 \Rightarrow \sec t = 0$ (impossible)

There are no points where the tangents are vertical.

**12. (a)** $\dfrac{d}{dt}(2\sin t) = 2\cos t$

$2\cos t = 0$

$t = \dfrac{\pi}{2}$ and $t = -\dfrac{\pi}{2}$

$y = 2\sin\left(\dfrac{\pi}{2}\right) = 2$ and $y = 2\sin\left(-\dfrac{\pi}{2}\right) = -2$

$x = -2\cos\left(\dfrac{\pi}{2}\right) = 0$ and $x = -2\cos\left(-\dfrac{\pi}{2}\right) = 0$

(0, 2) and (0, −2)

**(b)** $\dfrac{d}{dt}(2\cos t) = -2\sin t$

$-2\sin t = 0$

$t = 0$ and $t = \pi$

$x = -2\cos 0 = -2$ and $x = -2\cos\pi = 2$

$y = 2\sin 0 = 0$ and $y = 2\sin\pi = 0$

(−2, 0) and (2, 0)

**13. (a)** $\dfrac{d}{dt}(\cos^2 t) = -2\sin t\cos t$

$\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{-2\cos t\sin t}{\sin t} = -2\cos t$

$\dfrac{dy}{dx} = 0$ at (0, 0)

**(b)** $\dfrac{dx}{dy} = -\dfrac{1}{2\cos t}$ is never zero. There are no vertical tangents.

**14. (a)** $\dfrac{d}{dt}(9\sin t) = 9\cos t$

$9\cos t = 0$

$t = \dfrac{\pi}{2}$ and $t = -\dfrac{\pi}{2}$

$x = 4\cos\dfrac{\pi}{2} = 0$ $\quad x = 4\cos\left(-\dfrac{\pi}{2}\right) = 0$

$y = 9\sin\dfrac{\pi}{2} = 9$ $\quad y = 9\sin\left(-\dfrac{\pi}{2}\right) = -9$

(0, 9) and (0, −9)

**(b)** $\dfrac{d}{dt}(4\cos t) = -4\sin t$

$-4\sin t = 0$

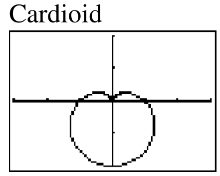
$t = 0$ or $t = \pi$

$x = 4\cos 0 = 4$, $x = 4\cos\pi = -4$

$y = 9\sin 0 = 0$, $y = 9\sin\pi = 0$

(4, 0) and (−4, 0)

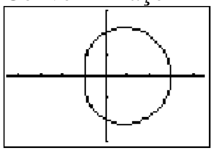
**15.** Cardioid



[−3, 3] by [−2, 2]

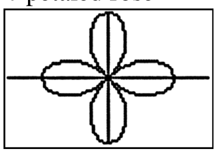
$0 \le \theta \le 2\pi$

**16.** Convex limaçon



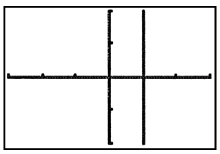
$[-4.5, 4.5]$ by $[-3, 3]$
$0 \le \theta \le 2\pi$

**17.** 4-petaled rose



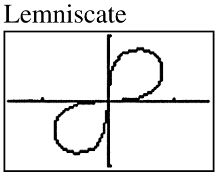
$[-1.5, 1.5]$ by $[-1, 1]$
$0 \le \theta \le 2\pi$

**18.** Vertical line



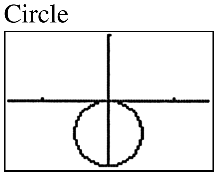
$[-3, 3]$ by $[-2, 2]$
$-\pi/2 \le \theta \le \pi/2$

**19.** Lemniscate



$[-1.5, 1.5]$ by $[-1, 1]$
$0 \le \theta \le \pi/2$

**20.** Circle



$[-1.5, 1.5]$ by $[-1, 1]$
$0 \le \theta \le \pi$

**21.** $$\frac{\frac{d}{d\theta}(\cos 2\theta)\sin\theta}{\frac{d}{d\theta}(\cos 2\theta)\cos\theta}$$
$$= \left[\frac{\cos\theta\cos 2\theta - 2\sin\theta\sin 2\theta}{-\sin\theta\cos 2\theta - 2\cos\theta\sin 2\theta}\right]_{\theta = \frac{\pi}{3}}$$
$$= 4.041$$

**22.** $$\frac{\frac{d}{d\theta}(2+\cos 2\theta)\sin\theta}{\frac{d}{d\theta}(2+\cos 2\theta)\cos\theta}$$
$$= \left[\frac{\cos\theta\cos 2\theta - 2\sin\theta\sin 2\theta + 2\cos\theta}{-\sin\theta\cos 2\theta - 2\cos\theta\sin 2\theta - 2\sin\theta}\right]_{\theta = \pi/3}$$
$$= 0.346$$

**23.** $$\frac{d}{d\theta}\left(1-\cos\left(\frac{\theta}{2}\right)\right)\sin\theta$$
$$= -\cos\theta\cos\left(\frac{\theta}{2}\right) + \frac{\sin\theta\sin\left(\frac{\theta}{2}\right)}{2} + \cos\theta$$
$$= 0$$
$y = 0, \quad y \approx \pm 0.443, \; y \approx \pm 1.739$
$$\frac{d}{d\theta}\left(1-\cos\left(\frac{\theta}{2}\right)\right)\cos\theta$$
$$= \sin\theta\cos\left(\frac{\theta}{2}\right) + \frac{\cos\theta\sin\left(\frac{\theta}{2}\right)}{2} - \sin\theta = 0$$
$x = 2, \; x \approx 0.067, \; x \approx -1.104$

**24.** $$\frac{d}{d\theta}(2(1-\sin\theta)\sin\theta) = (2-4\sin\theta)\cos\theta = 0$$
$$y = \frac{1}{2}, \; y = -4$$
$$\frac{d}{d\theta}(2(1-\sin\theta)\cos\theta)$$
$$= 2\sin\theta(\sin\theta - 1) - 2\cos^2\theta$$
$$= 0$$
$x = 0, \; x \approx \pm 2.598$

**25.** The tips of the petals are at the points where $r = \sin 2\theta = 1$, which are the points where $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$. The slope is
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dy}{d\theta}}$$
$$= \frac{\left(\frac{d}{d\theta}\right)(\sin 2\theta\sin\theta)}{\left(\frac{d}{d\theta}\right)(\sin 2\theta\cos\theta)}$$
$$= \frac{2\cos 2\theta\sin\theta + \sin 2\theta\cos\theta}{2\cos 2\theta\cos\theta + \sin 2\theta\sin\theta}.$$

| $\theta$ | $(x, y)$ | $m = \frac{dy}{dx}$ | Tangent line |
|---|---|---|---|
| $\frac{\pi}{4}$ | $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ | $-1$ | $y = -x + \sqrt{2}$ |
| $\frac{3\pi}{4}$ | $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ | $1$ | $y = x + \sqrt{2}$ |
| $\frac{5\pi}{4}$ | $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ | $-1$ | $y = -x - \sqrt{2}$ |
| $\frac{7\pi}{4}$ | $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ | $1$ | $y = x - \sqrt{2}$ |

**26.** The cardioid crosses the $x$-axis at the points where $\theta = 0$ and $\theta = \pi$. The slope is

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dy}{d\theta}}$$
$$= \frac{\left(\frac{d}{d\theta}\right)((1+\sin\theta)\sin\theta)}{\left(\frac{d}{d\theta}\right)((1+\sin\theta)\cos\theta)}$$
$$= \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta + (1+\sin\theta)\sin\theta}.$$

| $\theta$ | $(x, y)$ | $m = \frac{dy}{dx}$ | Tangent line |
|---|---|---|---|
| 0 | (1, 0) | 1 | $y = x - 1$ |
| $\pi$ | (–1, 0) | –1 | $y = -x - 1$ |

**27.** $x = y$, a line

**28.** $r^2 = 3r\cos\theta$

$x^2 + y^2 = 3x$, a circle

$\left(\text{center} = \left(\frac{3}{2}, 0\right), \text{radius} = \frac{3}{2}\right)$

**29.**
$$r^2 = 4r\frac{\sin\theta}{\cos\theta}\frac{1}{\cos\theta}$$
$$r^2\cos^2\theta = 4r\sin\theta$$
$$x^2 = 4y, \text{ a parabola}$$

**30.**
$$r\left(\cos\theta\cos\frac{\pi}{3} - \sin\theta\sin\frac{\pi}{3}\right) = 2\sqrt{3}$$
$$x\cos\frac{\pi}{3} - y\sin\frac{\pi}{3} = 2\sqrt{3}$$
$$\frac{1}{2}x - \frac{\sqrt{3}}{2}y = 2\sqrt{3}$$
$$x - \sqrt{3}y = 4\sqrt{3}$$
$$\text{or } y = \frac{x}{\sqrt{3}} - 4,$$

a line

**31.**
$$x^2 + y^2 + 5y = 0$$
$$r^2 + 5r\sin\theta = 0$$
$$r = -5\sin\theta$$

**32.**
$$x^2 + y^2 - 2y = 0$$
$$r^2 - 2r\sin\theta = 0$$
$$r = 2\sin\theta$$

**33.**
$$x^2 + 4y^2 = 16$$
$$r^2\cos^2\theta + 4r^2\sin^2\theta = 16, \text{ or}$$
$$r^2 = \frac{16}{\cos^2\theta + 4\sin^2\theta}$$

**34.**
$$(x+2)^2 + (y-5)^2 = 16$$
$$(r\cos\theta + 2)^2 + (r\sin\theta - 5)^2 = 16$$

**35.**
$$\int_0^{2\pi}\left(\frac{1}{2}(2-\cos 2\theta)^2\right)d\theta$$
$$= \left[\frac{1}{8}(\sin 2\theta\cos 2\theta - 2(4\sin 2\theta - 9\theta))\right]_0^{2\pi}$$
$$= \frac{9\pi}{2} - 0$$
$$= \frac{9\pi}{2}$$

**36.**
$$\frac{1}{3}\int_0^{\pi}\left(\frac{1}{2}(\sin 3\theta)^2\right)d\theta$$
$$= \left[-\frac{1}{36}(\sin 3\theta\cos 3\theta - 3\theta)\right]_0^{\pi}$$
$$= \frac{\pi}{12}$$

**37.** $4\left[\int_0^{\pi/4}\frac{1}{2}(1+\cos 2\theta)^2 d\theta - \int_0^{\pi/4}\frac{1}{2}(1)^2 d\theta\right] = 4\left[\int_0^{\pi/4}\left(\cos 2\theta + \frac{1+\cos 4\theta}{4}\right)d\theta\right]$

$$= 4\left[\frac{\sin 2\theta}{2} + \frac{4\theta + \sin 4\theta}{16}\right]_0^{\pi/4}$$

$$= \left[2\sin 2\theta + \theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi/4}$$

$$= 2 + \frac{\pi}{4}$$

**38.** $\int_0^{2\pi}\frac{1}{2}(2(1+\sin\theta))^2\, d\theta - \int_0^{\pi}\frac{1}{2}(2\sin\theta)^2\, d\theta = \int_0^{2\pi} 2(1+2\sin\theta+\sin^2\theta)\, d\theta - \int_0^{\pi} 2\sin^2\theta d\theta$

$$= \int_0^{2\pi}(2+4\sin\theta+1-\cos 2\theta)\, d\theta - \int_0^{\pi}(1-\cos 2\theta)\, d\theta$$

$$= \left[3\theta - 4\cos\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi} - \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi}$$

$$= 6\pi - 4 - 0 + 4 - \pi + 0$$

$$= 5\pi$$

**39.** **(a)** $\mathbf{v}(t) = \left\langle \frac{d}{dt}4\cos t, \frac{d}{dt}\sqrt{2}\sin t\right\rangle$

$\mathbf{v}(t) = \left\langle -4\sin t, \sqrt{2}\cos t\right\rangle$

$\mathbf{a}(t) = \left\langle \frac{d}{dt}(-4\sin t), \frac{d}{dt}\sqrt{2}\cos t\right\rangle$

$\mathbf{a}(t) = \left\langle -4\cos t, -\sqrt{2}\sin t\right\rangle$

**(b)** $\left\langle -4\cos\frac{\pi}{4}, -\sqrt{2}\sin\frac{\pi}{4}\right\rangle$

$\left\langle -4\frac{\sqrt{2}}{2}, -\sqrt{2}\frac{\sqrt{2}}{2}\right\rangle$

Speed $= \sqrt{\left(-2\sqrt{2}\right)^2 + (-1)^2}$

$= 3$

**40.** **(a)** $\mathbf{v}(t) = \left\langle \frac{d}{dt}\sqrt{3}\sec t, \frac{d}{dt}\sqrt{3}\tan t\right\rangle$

$\mathbf{v}(t) = \left\langle \sqrt{3}\sec t\tan t, \sqrt{3}\sec^2 t\right\rangle$

$\mathbf{a}(t) = \left\langle \frac{d}{dt}\sqrt{3}\sec t\tan t, \frac{d}{dt}\sqrt{3}\sec^2 t\right\rangle$

$\mathbf{a}(t) = \left\langle \sqrt{3}(\sec t\tan^2 t + \sec^3 t), 2\sqrt{3}\sec^2 t\tan t\right\rangle$

**(b)** $\left\langle \sqrt{3}\sec(0)\tan(0), \sqrt{3}\sec^2(0)\right\rangle$

$\left\langle 0, \sqrt{3}\right\rangle$

$$\text{Speed} = \sqrt{0^2 + \sqrt{3}^2} = \sqrt{3}$$

**41.** $\mathbf{v}(t) = \left\langle \frac{d}{dt}\frac{1}{\sqrt{1+t^2}}, \frac{d}{dt}\frac{t}{\sqrt{1+t^2}}\right\rangle$

$\mathbf{v}(t) = \left\langle \frac{-t}{(1+t^2)^{3/2}}, \frac{1}{(1+t^2)^{3/2}}\right\rangle$

$\text{speed} = \sqrt{\left(\frac{-t}{(1+t^2)^{3/2}}\right)^2 + \left(\frac{1}{(1+t^2)^{3/2}}\right)^2}$

$\text{speed} = \frac{1}{1+t^2}$

The maximum value of $\frac{1}{1+t^2}$ is 1, when $t = 0$.

**42.** $\mathbf{r}(t) = \left\langle e^t\cos t, e^t\sin t\right\rangle$, with slope

$\frac{e^t\sin t}{e^t\cos t} = \tan t.$

$\mathbf{v}(t) = \left\langle e^t\cos t - e^t\sin t, e^t\sin t + e^t\cos t\right\rangle$

$\mathbf{a}(t) = \left\langle -2e^t\sin t, 2e^t\cos t\right\rangle$, with slope

$\frac{2e^t\cos t}{-2e^t\sin t} = -\frac{1}{\tan t}$. Since the slopes are negative reciprocals, the angle is always 90°.

**43.** $\mathbf{r}(t) = \left\langle \int -\sin t\,dt, \int \cos t\,dt\right\rangle$

$\mathbf{r}(t) = \left\langle \cos t + C_1, \sin t + C_2\right\rangle$

$\cos(0) + C_1 = 0, C_1 = -1$

$\sin(0) + C_2 = 1, C_2 = 1$

$\mathbf{r}(t) = \langle \cos t - 1, \sin t + 1\rangle$

**44.** $\mathbf{r}(t) = \left\langle \int \frac{dt}{t^2+1}, \int \frac{t\,dt}{\left(t^2+1\right)^{1/2}}\right\rangle$

$\mathbf{r}(t) = \left\langle \tan^{-1} t + C_1, \sqrt{t^2+1} + C_2\right\rangle$

$\tan^{-1} 0 + C_1 = 1, C_1 = 1$

$\sqrt{0^2+1} + C_2 = 1, C_2 = 0$

$\mathbf{r}(t) = \left\langle \tan^{-1} t + 1, \sqrt{t^2+1}\right\rangle$

**45.** $\mathbf{v}(t) = \left\langle \int 0dt, \int 2dt\right\rangle$

$\mathbf{v}(t) = \langle 0, 2t + C\rangle$

$2(0) + C = 0$

$C = 0$

So $\mathbf{v}(t) = \langle 0, 2t\rangle$

$\mathbf{r}(t) = \left\langle \int 0\,dt, \int 2t\,dt\right\rangle$

$\mathbf{r}(t) = \langle 1, t^2 + C\rangle$

$0^2 + C = 0$

$C = 0$

$\mathbf{r}(t) = \left\langle 1, t^2\right\rangle$

**46.** $\mathbf{v}(t) = \left\langle \int -2\,dt, \int -2\,dt\right\rangle$

$\mathbf{v}(t) = \langle -2t + C_1, -2t + C_2\rangle$

$-2(1) + C_1 = 4, C_1 = 6$

$-2(1) + C_2 = 0, C_2 = 2$

$\mathbf{r}(t) = \left\langle \int (-2t+6)dt, \int (-2t+2)dt\right\rangle$

$\mathbf{r}(t) = \left\langle -t^2 + 6t + C_1, -t^2 + 2t + C_2\right\rangle$

$-(1)^2 + 6(1) + C_1 = 3, C_1 = -2$

$-(1)^2 + 2(1) + C_2 = 3, C_2 = 2$

$\mathbf{r}(t) = \left\langle -t^2 + 6t - 2, -t^2 + 2t + 2\right\rangle$

**47.** **(a)** $\frac{d}{dt}\left(3\cos\frac{\pi}{4}t\right) = \frac{-3\pi\sin\left(\frac{\pi}{4}t\right)}{4}$

$\frac{d}{dt}\left(5\sin\frac{\pi}{4}t\right) = \frac{5\pi\cos\left(\frac{\pi}{4}t\right)}{4}$

$$\sqrt{\left(\frac{-3\pi\sin\left(\frac{\pi}{4}t\right)}{4}\right)^2 + \left(\frac{5\pi\cos\left(\frac{\pi}{4}t\right)}{4}\right)^2}\,\Bigg|_{t=3}$$

$$= \frac{\pi\sqrt{17}}{4}$$

**(b)** $\dfrac{d}{dt}\left(\dfrac{-3\pi \sin\left(\frac{\pi}{4}t\right)}{4}\right)\Bigg|_{t=3}$

$= \dfrac{-3\pi^2 \cos\left(\frac{\pi}{4}t\right)}{16}\Bigg|_{t=3}$

$= \dfrac{3\pi^2}{16\sqrt{2}}$

$\dfrac{d}{dt}\left(\dfrac{5\pi \cos\left(\frac{\pi}{4}t\right)}{4}\right)\Bigg|_{t=3}$

$= -\dfrac{5\pi^2 \sin\left(\frac{\pi}{4}t\right)}{16}\Bigg|_{t=3}$

$= -\dfrac{5\pi^2}{16\sqrt{2}}$

**(c)** $x^2 = \left(3\cos\dfrac{\pi}{4}t\right)^2$ and $y^2 = \left(5\sin\dfrac{\pi}{4}t\right)^2$

$\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$

**48. (a)** $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$= \dfrac{e^t \sin t + e^t \cos t}{e^t \cos t - e^t \sin t}$

$= \dfrac{\cos t + \sin t}{\cos t - \sin t}$

$\dfrac{dy}{dx}\bigg|_{t=\pi} = \dfrac{-1}{-1} = 1$

**(b)** $\dfrac{dy}{dt} = e^t(\sin t + \cos t),\ \dfrac{dx}{dt} = e^t(\cos t - \sin t)$

$\left(\dfrac{dy}{dt}\right)^2 = e^{2t}(\sin^2 t + 2\sin t\cos t + \cos^2 t)$

$= e^{2t}(1 + 2\sin t\cos t)$

$\left(\dfrac{dx}{dt}\right)^2 = e^{2t}(\cos^2 t - 2\cos t\sin t + \sin^2 t)$

$= e^{2t}(1 - 2\cos t\sin t)$

$|\mathbf{v}(t)| = \sqrt{e^{2t}\cdot 2} = e^t\cdot\sqrt{2}$

$|\mathbf{v}(3)| = e^3\sqrt{2}$

**(c)** Distance $= \int_0^3 |\mathbf{v}(t)|\,dt$

$= \int_0^3 e^t\sqrt{2}$

$= \sqrt{2}[e^t]_0^3$

$= (e^3 - 1)\sqrt{2}$

**49. (a)** $\mathbf{v}(t) = \left\langle \dfrac{dx}{dt}, \dfrac{dy}{dt}\right\rangle = \left\langle 2t, \dfrac{6}{5}t^2\right\rangle,$

$\mathbf{v}(4) = \left\langle 8, \dfrac{96}{5}\right\rangle,$ and

$|\mathbf{v}(4)| = \sqrt{8^2 + \left(\dfrac{96}{5}\right)^2} = \dfrac{104}{5}$

**(b)** Distance $= \int_0^4 \sqrt{(2t)^2 + \left(\dfrac{6}{5}t^2\right)^2}\,dt$

$= \int_0^4 \dfrac{2}{5}t\sqrt{25 + 9t^2}\,dt$

$= \left[\dfrac{2}{135}(25 + 9t^2)^{3/2}\right]_0^4$

$= \dfrac{4144}{135}$

**(c)** $t = \sqrt{x+2}$, so

$\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dy}{dt}} = \dfrac{\frac{6t^2}{5}}{2t} = \dfrac{3}{5}t = \dfrac{3}{5}\sqrt{x+2}.$

**50.** $x$ degrees east of north is $(90 - x)$ degrees north of east.
Add the vectors:

$$\langle 540\cos 10°, 540\sin 10°\rangle + \langle 55\cos(-10°), 55\sin(-10°)\rangle = \langle 595\cos 10°, 485\sin 10°\rangle \approx \langle 585.961, 84.219\rangle.$$

Speed $\approx \sqrt{585.961^2 + 84.219^2} \approx 591.982$ mph.

Direction $\approx \tan^{-1}\left(\dfrac{585.961}{84.219}\right) \approx 81.821°$ east of north

**51.** **(a)** $\vec{a}_x = 2$

$\vec{v}_x = 2t + 0$

$x = t^2 + C \qquad x(0) = \pi \Rightarrow C = \pi$

$x = t^2 + \pi$

$y = \cos(t^2 + \pi)$

Position $= (t^2 + \pi, \cos(t^2 + \pi))$

**(b)** At this point $t^2 + \pi = 4$, so $t = \sqrt{4 - \pi}$.

$$\text{Speed} = \sqrt{(2t)^2 + (2t(-\sin(t^2+\pi))^2}\Big|_{t=\sqrt{4-\pi}} = 2.324$$

**52.** **(a)** $\mathbf{v}_A(t) = \langle 1, 2\rangle$ and $\mathbf{v}_B(t) = \left\langle \dfrac{3}{2}, \dfrac{3}{2}\right\rangle$

**(b)** $\int_0^3 \sqrt{1 + (2t-4)^2}\, dt \approx 6.126$

**(c)** Setting $x_A = x_B$, we find that $t = 4$. Plugging $t = 4$ into $y_A$ and $y_B$, we find that both values are the same (4). Thus, the particles collide when $t = 4$. (*Note*: If you graph both paths, they will cross at (–1, 1). However, the particles are there at different times.

**53.** **(a)** $\text{Area} = \int_0^\pi \dfrac{1}{2}\left(\dfrac{4}{1+\sin\theta}\right)^2 d\theta = \dfrac{32}{3}$

**(b)** The polar equation is equivalent to $r + r\sin\theta = 4$. Thus,

$$\begin{aligned} r &= 4 - r\sin\theta \\ r^2 &= (4 - r\sin\theta)^2 \\ x^2 + y^2 &= (4 - y)^2 \\ x^2 + y^2 &= 16 - 8y + y^2 \\ 8y &= 16 - x^2 \end{aligned}$$

**(c)** $\text{Area} = \int_{-4}^{4}\left(2 - \dfrac{x^2}{8}\right) dx$, which, indeed, is $\dfrac{32}{3}$.